University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
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Solution Exam I

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Problems

1. [20 points] For what values of t are the pair of vectors $\langle t, -1, t-1\rangle$ and $\langle -3, -1, -4\rangle$ parallel?

   Solution: If vector $\langle t, -1, t-1\rangle$ is parallel to vector $\langle -3, -1, -4\rangle$, we can write:

   $\alpha\langle t, -1, t-1\rangle = \langle -3, -1, -4\rangle$

   Which means that:

   (1) $\alpha t = -3$
   (2) $\alpha(-1) = -1$
   (3) $\alpha(t-1) = -4$

   From Eq. (2), we see that $\alpha = 1$. Substituting this value in Eq. (1), we obtain $t = -3$. Now, since the three equations must be satisfied simultaneously, we test these values in Eq. (3). Doing so, we get $-3-1 = -4$, which is true. Therefore, the value of t that makes $\langle t, -1, t-1\rangle$ and $\langle -3, -1, -4\rangle$ parallel, is $t = -3$.

2. [20 points] If $\vec{a} = \langle 12, -4, 3\rangle$, $\vec{b} = \langle 4, -3, 0\rangle$, and θ is the angle between them, find $\tan\theta$.

   Solution: Given that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, then $\tan\theta = \frac{||\vec{a}\times\vec{b}||}{\vec{a}\cdot\vec{b}}$. This is because $||\vec{a}\times\vec{b}|| = ||\vec{a}||\,||\vec{b}||\sin\theta$ and $\vec{a}\cdot\vec{b} = ||\vec{a}||\,||\vec{b}||\cos\theta$.

In this case, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 12 & -4 & 3 \\ 4 & -3 & 0 \end{vmatrix} = 9\hat{i} + 12\hat{j} - 20\hat{k} = \langle 9, 12, -20 \rangle.$$

And therefore $||\vec{a} \times \vec{b}|| = \sqrt{81 + 144 + 400} = \sqrt{625} = 25$.

$\vec{a} \cdot \vec{b} = \langle 12, -4, 3 \rangle \cdot \langle 4, -3, 0 \rangle = 48 + 12 = 60$.

We conclude then that $\tan\theta = \frac{||\vec{a}\times\vec{b}||}{\vec{a}\cdot\vec{b}} = \frac{25}{60} = \frac{5}{12}$.

3. [20 points] Find a unit vector that is perpendicular to both $\vec{a} = \langle 2, -1, 0 \rangle$ and $\vec{b} = \langle -1, 1, 4 \rangle$.

Solution: A vector that satisfies the give requirements is $\hat{u} = \frac{\vec{a}\times\vec{b}}{||\vec{a}\times\vec{b}||}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 0 \\ -1 & 1 & 4 \end{vmatrix} = -4\hat{i} - 8\hat{j} + \hat{k} = \langle -4, -8, 1 \rangle.$$

$||\vec{a} \times \vec{b}|| = \sqrt{16 + 64 + 1} = \sqrt{81} = 9$.

Therefore, $\hat{u} = \frac{1}{9}\langle -4, -8, 1 \rangle = \langle -\frac{4}{9}, -\frac{8}{9}, \frac{1}{9} \rangle$.

4. [20 points] Find the area of the triangle with vertices $(1, 1, 1)$, $(0, 1, 0)$, and $(1, 0, 1)$.

Solution: If $A(1, 1, 1)$, $B(0, 1, 0)$, and $C(1, 0, 1)$, then the area of the triangle ABC is given by $a = \frac{1}{2}||\vec{AB} \times \vec{AC}||$.

So, $\vec{AB} = \langle -1, 0, -1 \rangle$ and $\vec{AC} = \langle 0, -1, 0 \rangle$ and

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix} = -\hat{i} + \hat{k} = \langle -1, 0, 1 \rangle.$$

And $a = \frac{1}{2}||\vec{AB} \times \vec{AC}|| = \frac{1}{2}\sqrt{2}$.

5. [20 points] Consider points $A(2, 1, 3)$, $B(1, 2, 1)$, $C(-1, -1, -2)$, and $D(1, -4, 0)$. Find the shortest distance between the line that passes through points A and B, and the line that passes through points C and D.

Solution: We have seen in class and in homeworks that this distance is $d = \left|\frac{\vec{PP}\cdot(\vec{v}_1\times\vec{v}_2)}{||\vec{v}_1\times\vec{v}_2||}\right|$, where $\vec{PP}$ is a vector that joins a point on one line to a point on the other line, and $\vec{v}_1$ and $\vec{v}_2$ are the direction vectors of the lines involved in the problem.

In our case, the direction vector of the line through A and B is $\vec{v}_1 = \langle -1, 1, -2 \rangle$, and the direction vector of the line through C and D is $\vec{v}_2 = \langle 2, -3, 2 \rangle$. The vector $\vec{PP}$ may be the vector that joins A and C, so

$\vec{PP} = \langle -3, -2, -5 \rangle$. Then,

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 2 & -3 & 2 \end{vmatrix} = -4\hat{i} - 2\hat{j} + \hat{k} = \langle -4, -2, 1 \rangle$$

So, $||\vec{v}_1 \times \vec{v}_2|| = \sqrt{16+4+1} = \sqrt{21}$.

Therefore, $d = \left| \frac{1}{\sqrt{21}} (\langle -3, -2, -5 \rangle \cdot \langle -4, -2, 1 \rangle) \right| = \left| \frac{1}{\sqrt{21}} (12 + 4 - 5) \right| = \left| \frac{1}{\sqrt{21}} (11) \right| = \frac{11}{\sqrt{21}}$.

[Bonus problem: 10 points] If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors, does $\vec{a} \times \vec{b} = \vec{c} \times \vec{b}$ imply that $\vec{a} = \vec{c}$? Show that it is true in general, or disprove by providing an appropriate example.

Solution: No, that $\vec{a} \times \vec{b} = \vec{c} \times \vec{b}$ does not imply that $\vec{a} = \vec{c}$.

The cross product $\vec{a} \times \vec{b}$ will produce a vector perpendicular to both with a magnitude equal to $||\vec{a}|| ||\vec{b}|| \sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$. If another vector $\vec{c}$ is coplanar to $\vec{a}$ and $\vec{b}$ (that is, they lie in the same plane), then the cross product $\vec{c} \times \vec{b}$ will produce a vector perpendicular not only to $\vec{c}$ and $\vec{b}$, but also to $\vec{a}$. Then, if $\vec{a} \times \vec{b} = \vec{c} \times \vec{b}$, their magnitudes must be equal too, so $||\vec{a} \times \vec{b}|| = ||\vec{c} \times \vec{b}|| \rightarrow ||\vec{a}|| ||\vec{b}|| \sin\theta = ||\vec{c}|| ||\vec{b}|| \sin\alpha$, where α is the angle between $\vec{c}$ and $\vec{b}$. This means that vectors $\vec{a}$ and $\vec{c}$ will produce the same cross product with $\vec{b}$ as long as $||\vec{a}|| \sin\theta = ||\vec{c}|| \sin\alpha$. This situation is illustrated below.

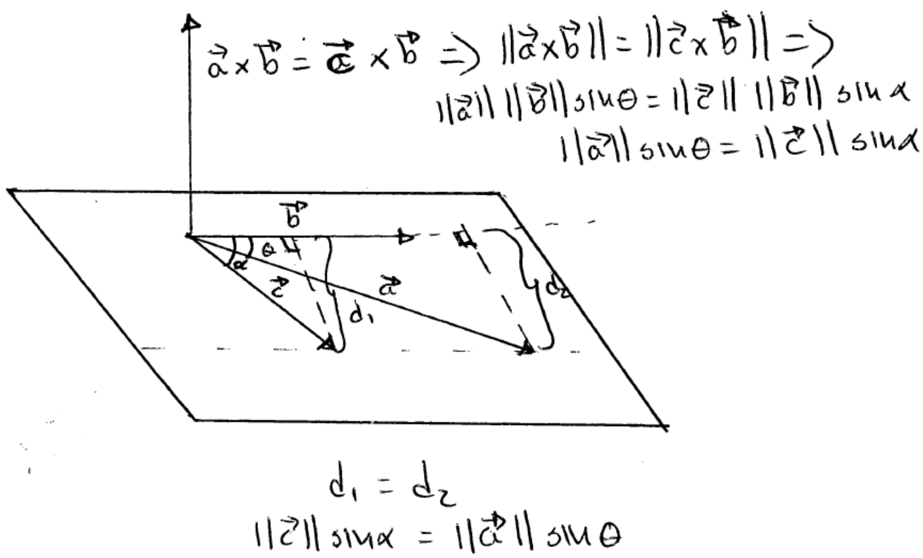

Figure 1: That $\vec{a} \times \vec{b} = \vec{c} \times \vec{b}$ does not imply that $\vec{a} = \vec{c}$.