University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
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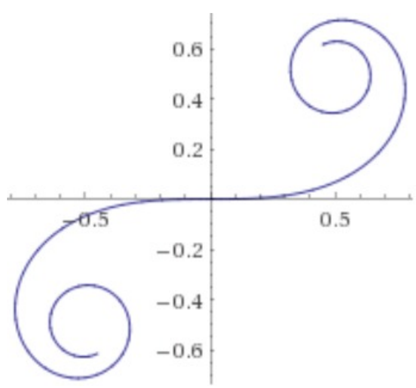
Fall 2012

Solution Exam II

October 24, 2012

Problems

1. [20 points] A **cornu spiral** is given by $\vec{r}(t) = \left\langle \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du, \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du \right\rangle$. The spiral shown in the figure was plotted over the interval $-2.5 \leq t \leq 2.5$.



a) [9 points] Find the arc length of this curve from $t = 0$ to $t = a$. [Hint: The Fundamental Theorem of Calculus may help you a lot here.]

b) [9 points] Find the curvature of the graph when $t = a$.

c) [2 points] The cornu spiral was discovered by James Bernoulli. He found that the spiral has a very interesting relationship between curvature and arc length. What is the relationship?

Solution:

a) The arc length of a curve represented by a vector function $\vec{r}(t)$ is given by $\int_{t=t_0}^{t=t_1} ||\vec{r}\,'(t)||\, dt$. So, differentiating $\vec{r}(t)$, we obtain $\vec{r}\,'(t) = \left\langle \cos\left(\frac{\pi t^2}{2}\right), \sin\left(\frac{\pi t^2}{2}\right) \right\rangle$ using the Fundamental Theorem of Calculus.

Then, $||\vec{r}\,'(t)|| = \sqrt{\cos^2\left(\frac{\pi t^2}{2}\right) + \sin^2\left(\frac{\pi t^2}{2}\right)} = \sqrt{1} = 1$. Thus, $\int_{t=0}^{t=a} dt = a$.

b) The curvature of $\vec{r}(t)$ can be calculated using $\kappa = \frac{||\vec{r}\,'(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3}$. Therefore, differentiating $\vec{r}\,'(t)$ we obtain $\vec{r}\,''(t) = \left\langle -\pi t \sin\left(\frac{\pi t^2}{2}\right), \pi t \cos\left(\frac{\pi t^2}{2}\right) \right\rangle$, and thus $\vec{r}\,'(t) \times \vec{r}\,''(t) = \pi t \hat{k}$, which implies that $||\vec{r}\,'(t) \times \vec{r}\,''(t)|| = \pi|t| = \pi t$, if we assume that $t \geq 0$.

The curvature $\kappa = \frac{\pi t}{1^3} = \pi t$, and at $t = a$, $\kappa = a\pi$.

c) The relationship between curvature κ and arc length s for a cornu spiral is therefore $\kappa = \pi s$.

2. [20 points] Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

Solution:

We need to find a vector function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. Since $z = xy$, it is clear that the function we are looking for will be given by $\vec{r}(t) = \langle x(t), y(t), x(t)y(t) \rangle$. Thus, the problem is reduced to find parametric equations for x and y.

Since the cross section of the cylinder parallel to the xy-plane is a circle of radius 2, then we can choose $x = 2\cos t$ and $y = 2\sin t$.

Therefore, the vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$ can be represented by $\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\sin t\cos t \rangle$.

3. [20 points] Find the position vector of a particle whose acceleration is given by $\vec{a}(t) = \langle 2t, \sin t, \cos 2t \rangle$, initial velocity $\vec{v}(0) = \hat{\imath}$ and initial position $\vec{r}(0) = \hat{\jmath}$.

Solution:

The velocity of the particle $\vec{v}(t)$ can be found by integrating the acceleration with respect to time. Thus, $\vec{v}(t) = \int \vec{a}(t)\, dt = \langle t^2, -\cos t, \frac{1}{2}\sin 2t \rangle + \vec{A}$, where $\vec{A}$ is a constant vector.

Since $\vec{v}(0) = \langle 1, 0, 0 \rangle$, then $\vec{v}(0) = \langle 0, -1, 0 \rangle + \vec{A} = \langle 1, 0, 0 \rangle$. This implies that $\vec{A} = \langle 1, 1, 0 \rangle$, and therefore that $\vec{v}(t) = \langle t^2 + 1, 1 - \cos t, \frac{1}{2}\sin 2t \rangle$.

The position vector of the particle can be found by integrating the velocity with respect to time. Thus, $\vec{r}(t) = \int \vec{v}(t)\, dt = \langle \frac{t^3}{3} + t, t - \sin t, -\frac{1}{4}\cos 2t \rangle + \vec{B}$, where $\vec{B}$ is a constant vector.

Since $\vec{r}(0) = \langle 0, 1, 0 \rangle$, we have that $\vec{r}(0) = \langle 0, 0, -\frac{1}{4} \rangle + \vec{B} = \langle 0, 1, 0 \rangle$. This means that $\vec{B} = \langle 0, 1, \frac{1}{4} \rangle$ and $\vec{r}(t) = \langle \frac{t^3}{3} + t, t - \sin t + 1, \frac{1}{4} - \frac{1}{4}\cos 2t \rangle$.

4. [20 points] Find the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$.

Solution: Directly substituting values we obtain an indeterminate form, $\frac{0}{0}$. However, noting that the numerator is a difference of squares, we can factorize the numerator as $(x-y)(x+y)$ and therefore:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(x+y)}{x+y} = \lim_{(x,y)\to(0,0)} (x-y) = 0.$$

5. [20 points] Find **all the second order partial derivatives** of $f(x,y) = \frac{xy}{x+1}$.

Solution:

$$\frac{\partial f}{\partial x} = y\left(\frac{\partial}{\partial x}\left(\frac{x}{x+1}\right)\right) = y\left(\frac{(x+1)(1)-x(1)}{(x+1)^2}\right) = y\left(\frac{1}{(x+1)^2}\right) = \frac{y}{(x+1)^2}.$$

$$\frac{\partial f}{\partial y} = \left(\frac{x}{x+1}\right)\frac{\partial}{\partial y}y = \frac{x}{x+1}.$$

$$\frac{\partial^2 f}{\partial x^2} = y\left(\frac{\partial}{\partial x}\left(\frac{1}{(x+1)^2}\right)\right) = y\left(\frac{(x+1)^2(0)-(1)(2(x+1)(1))}{(x+1)^4}\right) = y\frac{-2(x+1)}{(x+1)^4} = \frac{-2y}{(x+1)^3}.$$

$$\frac{\partial^2 f}{\partial y^2} = 0.$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{1}{(x+1)^2}\frac{\partial}{\partial y}y = \frac{1}{(x+1)^2}.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x+1)(1)-x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}.$$

[Bonus problem: 10 points] Find an equation of the tangent plane to $z = e^{\sqrt{xy}}$ at $(1,1,e)$.

Solution:

$f_x(x,y) = e^{\sqrt{xy}}\left(\frac{1}{2}(xy)^{-1/2}(y)\right) = \frac{ye^{\sqrt{xy}}}{2\sqrt{xy}}$.

$f_y(x,y) = e^{\sqrt{xy}}\left(\frac{1}{2}(xy)^{-1/2}(x)\right) = \frac{xe^{\sqrt{xy}}}{2\sqrt{xy}}$.

The tangent plane's equation at $(1,1,e)$ is therefore:

$f_x(1,1)(x-1) + f_y(x,y)(y-1) - (z-e) = 0$
$\frac{e}{2}(x-1) + \frac{e}{2}(y-1) - z + e = 0$
$\frac{e}{2}x - \frac{e}{2} + \frac{e}{2}y - \frac{e}{2} - z + e = 0$
$\frac{e}{2}x + \frac{e}{2}y - z = 0$ or $ex + ey - 2z = 0$.