University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
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Fall 2012

Solution Exam III

November 26, 2012

Problems

1. [20 points] A particle is traveling along the path given by the following parametric equations: $x = 2 - t$ and $y = 2 - t^2$. At what rate is the distance between the origin and the particle changing when $t = 1$?

   Solution: The distance between the origin and the particle is $w = \sqrt{x^2 + y^2}$. Therefore,

   $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$.

   In our case:
   $\frac{\partial w}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$, which in terms of t and at $t = 1$ becomes $\left.\frac{2-t}{\sqrt{(2-t)^2+(2-t^2)^2}}\right|_{t=1} = \frac{1}{\sqrt{(1)^2+(1)^2}} = \frac{1}{\sqrt{2}}$
   similarly, $\frac{\partial w}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{2}}$ at $t = 1$
   $\frac{dx}{dt} = -1$
   $\frac{dy}{dt} = -2t$, which at $t = 1$ becomes $\frac{dy}{dt} = -2$.

   We conclude then that $\frac{dw}{dt} = \frac{1}{\sqrt{2}}(-1) + \frac{1}{\sqrt{2}}(-2) = -\frac{3}{\sqrt{2}}$.

2. [20 points] Marine biologists have determined that when a shark detects the presence of blood in the water, it will swim in the direction in which the concentration of the blood increases most rapidly. Based on certain tests, the concentration of blood (in parts per million) at a point $P(x, y, z)$ in the water is given by

   $$C(x, y, z) = e^{-(x^2+2y^2+3z^2)/10^4}$$

   where x, y, z are measured in meters in a rectangular coordinate system with the blood source at the origin. Suppose a shark is located at $(1, 2, 0)$ when it first detects the presence blood in the water. In which direction will it start swimming? [The answer should be a vector that gives the shark's direction of movement from its position.]

Solution: At any point in the domain of a function, the gradient points in the direction in which a function increases most rapidly. So, the problem in this case is simply to find the direction in which the gradient of $C(x, y, z)$ is pointing.

$\nabla C(x, y, z) = \langle e^{-(x^2+2y^2+3z^2)/10^4}(-2x/10^4), e^{-(x^2+2y^2+3z^2)/10^4}(-4y/10^4), e^{-(x^2+2y^2+3z^2)/10^4}(-6z/10^4)\rangle$
$= e^{-(x^2+2y^2+3z^2)/10^4}/10^4\langle -2x, -4y, -6z\rangle$

Evaluating $\nabla C(x, y, z)$ at the shark's location: $\nabla C(1, 2, 0) = e^{-9/10^4}/10^4\langle -2, -8, 0\rangle$.

Therefore, we conclude that the shark will swim in the direction of the vector $\langle -2, -8, 0\rangle$.

3. [20 points] Find any extrema (i.e., either a maximum or a minimum) of $f(x, y) = x^2 + 3xy + y^2$ subject to the constraint $x^2 + y^2 = 1$.

   Solution: Let $g(x, y) = x^2 + y^2 - 1$, so points at which $f(x, y) = x^2 + 3xy + y^2$ is maximum or minimum subject to $g(x, y) = 0$ will satisfy:

   $\nabla f(x, y) = \lambda \nabla g(x, y)$
   $\langle 2x + 3y, 2y + 3x\rangle = \lambda\langle 2x, 2y\rangle$
   So,
   $2x + 3y = 2\lambda x$ (1)
   $2y + 3x = 2\lambda y$ (2)
   $x^2 + y^2 = 1$ (3)

   From (1) and (2):
   $\lambda = \frac{2x+3y}{2x} = \frac{2y+3x}{2y}$
   $(2y)(2x + 3y) = (2x)(2y + 3x)$
   $4xy + 6y^2 = 4xy + 6x^2$
   $y^2 = x^2$ (4)

   (4) in (3):
   $x^2 + x^2 = 2x^2 = 1$, which means that $x = \pm\frac{1}{\sqrt{2}}$ and from (4), $y = \pm\frac{1}{\sqrt{2}}$. So candidate extrema are found at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

   Evaluating the function at these points: $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{5}{2}$ which are maxima, and $f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\frac{1}{2}$, which are minima.

4. [20 points] Use a double integral in polar coordinates to find the volume of a sphere of radius a.

   Solution: Let us denote by V the volume of the sphere of radius a. We will find the volume of the upper half hemisphere and multiply at the end by 2. Thus,

   $\frac{V}{2} = \iint_D \sqrt{a^2 - x^2 - y^2}\, dA$, where D is the disk enclosed by $x^2 + y^2 = a^2$.

   In polar coordinates:

$\frac{V}{2} = \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}\, r\, dr\, d\theta$. Using $u = a^2 - r^2$, $du = -2r\, dr$ and therefore

$\frac{V}{2} = \frac{-1}{2} \int_0^{2\pi} \int_{a^2}^0 \sqrt{u}\, du\, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^{a^2} \sqrt{u}\, du\, d\theta$

$\frac{V}{2} = \frac{1}{2} \int_0^{2\pi} \frac{2}{3} u^{3/2}\Big|_0^{a^2} d\theta = \frac{1}{3} \int_0^{2\pi} a^3\, d\theta = \frac{a^3}{3}(2\pi) = \frac{2\pi a^3}{3}$.

Therfore, $V = \frac{4\pi a^3}{3}$.

5. [20 points] Use spherical coordinates to find the average distance from a point in a ball of radius a to its center. Remember that the average value of a function of three variables over a solid region E is defined as $f_{\text{avg}} = \dfrac{1}{V(E)} \iiint_E f(x, y, z)\, \mathrm{d}V$, where $V(E)$ is the volume of E.

Solution: Placing the center of the ball at the origin, the distance from a point with coordinates (x, y, z) to the center of the ball is $d = \sqrt{x^2 + y^2 + z^2}$, which in spherical coordinates is $d = \rho$.

Then, the $d_{\text{avg}} = \frac{1}{\frac{4\pi a^3}{3}} \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$

$d_{\text{avg}} = \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta$

$d_{\text{avg}} = \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi} \frac{\rho^4}{4}\Big|_0^a \sin\phi\, d\phi\, d\theta$

$d_{\text{avg}} = \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi} \frac{a^4}{4} \sin\phi\, d\phi\, d\theta$

$d_{\text{avg}} = \frac{3a}{16\pi} \int_0^{2\pi} \int_0^{\pi} \sin\phi\, d\phi\, d\theta$

$d_{\text{avg}} = \frac{3a}{16\pi} \int_0^{2\pi} -\cos\phi|_0^{\pi}\, d\phi\, d\theta$

$d_{\text{avg}} = \frac{3a}{16\pi} \int_0^{2\pi} 2\, d\theta$

$d_{\text{avg}} = \frac{3a}{8\pi} \int_0^{2\pi} d\theta$

$d_{\text{avg}} = \frac{3a}{8\pi}(2\pi) = \frac{3a}{4}$

[Bonus problem: 10 points] Find and classify all the critical points of $f(x, y) = x^3 - 12xy + 8y^3$.

Solution: $f_x(x, y) = 3x^2 - 12y = 0$ (1) and $f_y(x, y) = -12x + 24y^2 = 0$ (2). From (1) $x^2 = 4y$ and from (2) $x = 2y^2$. Substituting the second equation into the first one: $(2y^2)^2 = 4y$, so $4y^4 - 4y = 0$ or $y(y^3 - 1) = 0$, so $y = 0$ or $y = 1$. Therefore, the critical points are:

$(0, 0)$, and $(2, 1)$.

Now, $f_{xx}(x, y) = 6x$, $f_{xy}(x, y) = -12$, and $f_{yy}(x, y) = 48y$ and then $\det(H)(x, y) = (6x)(48y) - (-12)^2 = 288xy - 144$.

For $(0, 0)$, $\det(H)(0, 0) = -144 < 0$, and therefore at $(0, 0)$, the function has a saddle point.
For $(2, 1)$, $\det(H)(2, 1) = 576 - 144 > 0$ and $f_{xx}(2, 1) > 0$, so at $(2, 1)$, the function has a local minimum.