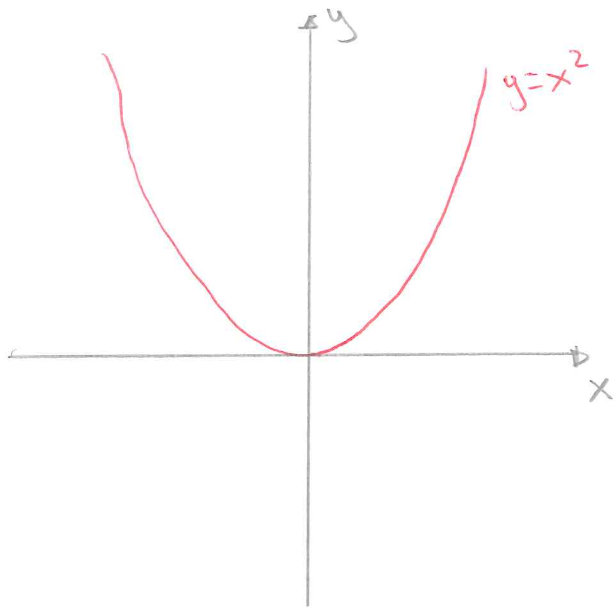
Homework #1

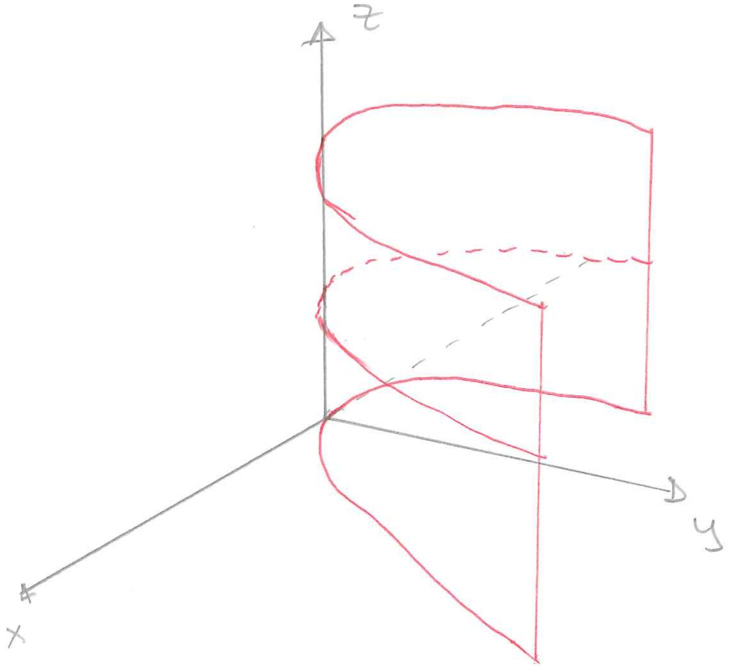
MATH 243 - Section 50

Dr. Marco A. Montes de Oca

1. a) The equation $y=x^2$ represents a parabola in $\mathbb{R}^2$.



b) In $\mathbb{R}^3$, $y=x^2$ represents a surface traced out by lines parallel to the z-axis that follow a parabola in the xy-plane. Technically speaking, it is a parabolic cylinder.



2. The distance from a point to a geometric object is usually understood to be the shortest distance between them.

a) In this case, the shortest distance between $(1,1,-5)$ and the xy-plane is simply $|-5| = 5$. In other wards, it is the distance between $(1,1,-5)$ and $(1,1,0)$ which lies on the xy-plane.

b) The distance between two points is given by $d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$. In this case we have:

$$d = \sqrt{(1-0)^2 + (1-1)^2 + (-5-10)^2} = \sqrt{1+225} = \sqrt{226}$$

c) Using the same reasoning as in (a), we have that the distance between $(1,1,-5)$ and the plane $z=2$ is equal to $\underline{|-5-2| = 7}$

d) Similarly, the distance between $(1,1,-5)$ and the x-axis is equal to the distance between $(1,1,-5)$ and $(1,0,0)$, which is equal to $\sqrt{(1-1)^2+(0-1)^2+(0+5)^2} = \sqrt{1+25} = \underline{\sqrt{26}}$

e) The plane $z=0$ is the xy-plane, so the answer is the same as in a), that is, $\underline{|-5| = 5}$

f) This distance is equal to $\sqrt{(0-1)^2+(0-1)^2+(-5+5)^2}$
$\boxed{= \sqrt{2}}$

3. The distance between $(6,-2,3)$ and $(-1,2,1)$ should be equal to the radius of the sphere we are looking for. Therefore:

$$r = \sqrt{(6-(-1))^2+(-2-2)^2+(3-1)^2}$$
$$= \sqrt{49+16+4} = \sqrt{69}$$

The equation of the sphere is then

$$(x+1)^2+(y-2)^2+(z-1)^2=\sqrt{69}$$

---

4. If $\vec{r}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$

$$=\alpha\langle 2,-1,1\rangle+\beta\langle 1,-3,-2\rangle+\gamma\langle -2,1,-3\rangle$$

$$=\langle 2\alpha,-\alpha,\alpha\rangle+\langle \beta,-3\beta,-2\beta\rangle+\langle -2\gamma,\gamma,-3\gamma\rangle$$

$$=\langle 2\alpha+\beta-2\gamma,-\alpha-3\beta+\gamma,\alpha-2\beta-3\gamma\rangle=\langle 1,3,2\rangle$$

Thus:

$$\begin{cases} 2\alpha+\beta-2\gamma=1 & (1) \\ -\alpha-3\beta+\gamma=3 & (2) \\ \alpha-2\beta-3\gamma=2 & (3) \end{cases}$$

(2) + (3):

$$-5\beta-2\gamma=5 \quad (4)$$

2 (2) + 1:

$$-5\beta=7 \Rightarrow \underline{\beta=-\frac{7}{5}} \quad (5)$$

(5) in (4):

$$-5\left(-\frac{7}{5}\right)-2\gamma=5$$

$$7-2\gamma=5 \Rightarrow \underline{\gamma=1} \quad (6)$$

(5) and (6) in (3):

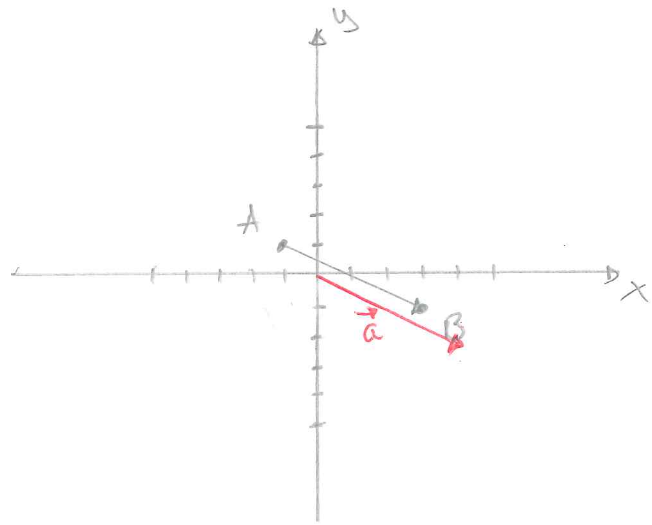
$$\alpha=2+2\beta+3\gamma$$

$$=2+2\left(-\frac{7}{5}\right)+3(1)$$

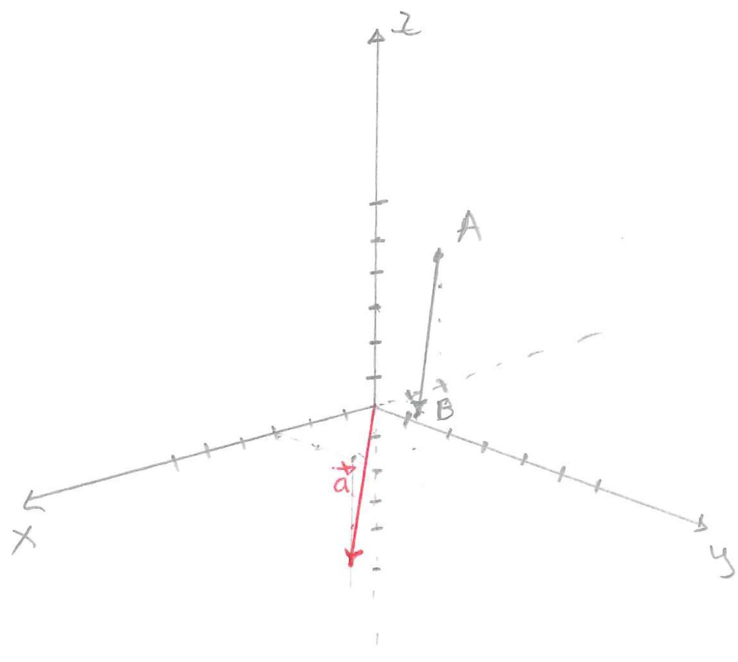
$$=5-\frac{14}{5}=\frac{25-14}{5}$$

$$=\underline{\frac{11}{5}}$$

5. a) $\vec{a} = \langle 3-(-1), -1-1 \rangle = \langle 4, -2 \rangle$



b) $\vec{a} = \langle 1-(-2), 2-0, 1-4 \rangle = \langle 3, 2, -3 \rangle$



6.

a) $2\vec{a} - \vec{b} + 3\vec{c} = 2\langle 3,-1,-4\rangle - \langle -2,4,-3\rangle + 3\langle 1,2,-1\rangle$

$= \langle 6,-2,-8\rangle + \langle 2,-4,3\rangle + \langle 3,6,-3\rangle$

$= \underline{\langle 11, 0, -8\rangle}$

b) $\|\vec{a}+\vec{b}+\vec{c}\| = \|\langle 3,-1,-4\rangle + \langle -2,4,-3\rangle + \langle 1,2,-1\rangle\|$

$= \|\langle 2,5,-8\rangle\| = \sqrt{2^2+5^2+(-8)^2}$

$= \sqrt{4+25+64} = \underline{\sqrt{93}}$

c) $\|3\vec{a}-2\vec{b}+4\vec{c}\| = \|3\langle 3,-1,-4\rangle - 2\langle -2,4,-3\rangle + 4\langle 1,2,-1\rangle\|$

$= \|\langle 9,-3,-12\rangle + \langle 4,-8,6\rangle + \langle 4,8,-4\rangle\|$

$= \|\langle 17,-3,-10\rangle\| = \sqrt{17^2+(-3)^2+(-10)^2}$

$= \sqrt{289+9+100}$

$= \underline{\sqrt{398}}$

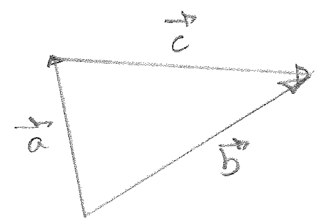
d) If $\vec{r} = 3\vec{a}-2\vec{b}+4\vec{c}$, then

$$\underline{\hat{r} = \frac{\vec{r}}{\|\vec{r}\|} = \frac{1}{\sqrt{398}}\langle 17,-3,-10\rangle}$$

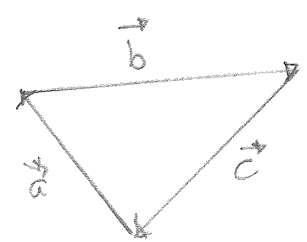
7. Three vectors form the sides of a triangle if their signed sum is equal to the zero vector. In mathematical terms, if we label the vectors as $\vec{a}$, $\vec{b}$, and $\vec{c}$, then

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$$

where $\alpha, \beta, \gamma \in \{-1, 1\}$. For example, it may be that $\vec{a} - \vec{b} + \vec{c} = \vec{0}$, or $\vec{a} + \vec{c} = \vec{b}$ as illustrated below



or $\quad \vec{a} + \vec{b} + \vec{c} = \vec{0}$



In this particular example:

$$\alpha\langle 3,-1,-2\rangle + \beta\langle -1,3,4\rangle + \gamma\langle 4,-2,-6\rangle = \vec{0}$$

$$\langle 3\alpha - \beta + 4\gamma,\ -\alpha + 3\beta - 2\gamma,\ -2\alpha + 4\beta - 6\gamma\rangle = \langle 0,0,0\rangle$$

Therefore:

$$\begin{cases} 3\alpha - \beta + 4\gamma = 0 \\ -\alpha + 3\beta - 2\gamma = 0 \\ -2\alpha + 4\beta - 6\gamma = 0 \end{cases}$$

Solving the system we find that (we discard the solution $\alpha = \beta = \gamma = 0$).

$\beta = \gamma$ and $\alpha = -\gamma$

Since α, β and $\gamma \in \{-1, 1\}$ there are only two possible solutions:

$$\beta = \gamma = 1, \ \alpha = -1$$

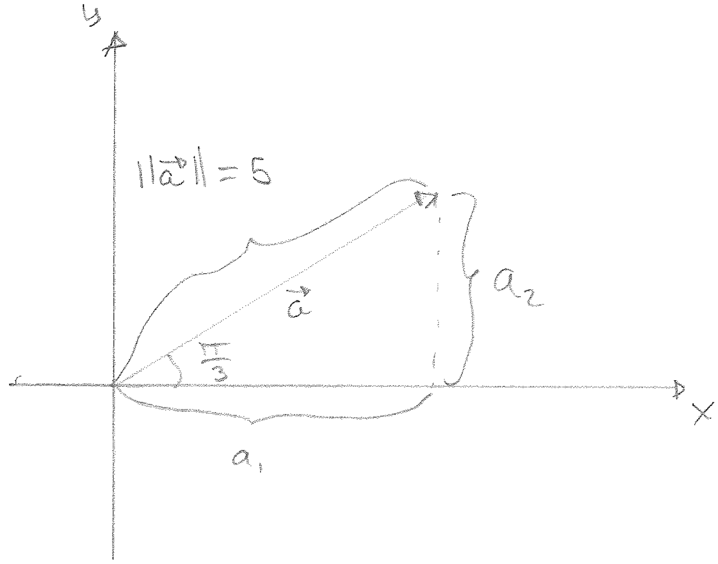
or

$$\beta = \gamma = -1, \ \alpha = 1$$

And thus we can write $-\vec{a} + \vec{b} + \vec{c} = \vec{0}$, or

$\underline{\vec{b} + \vec{c} = \vec{a}}$

8.

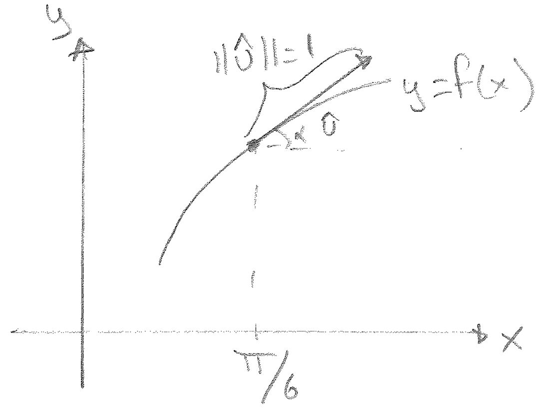


$a_1 = \|\vec{a}\| \cos\left(\frac{\pi}{3}\right) = 5\left(\frac{1}{2}\right) = \frac{5}{2}$

$a_2 = \|\vec{a}\| \sin\left(\frac{\pi}{3}\right) = 5\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2}$

$\therefore \underline{\vec{a} = \left\langle \frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle}$

9.



Slope at $\pi/6$: $m = y'\big|_{x=\pi/6} = 2\cos\left(\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

By definition: $m = \tan\alpha = \frac{\text{y-component}}{\text{x-component}} = \frac{b}{a}$ ①

Additionally: $\sqrt{a^2 + b^2} = 1$ (unit vector) ② then $\sqrt{3}a = b \Rightarrow$

$$\sqrt{a^2 + (\sqrt{3}a)^2} = 1$$

$$\sqrt{a^2 + 3a^2} = 1$$

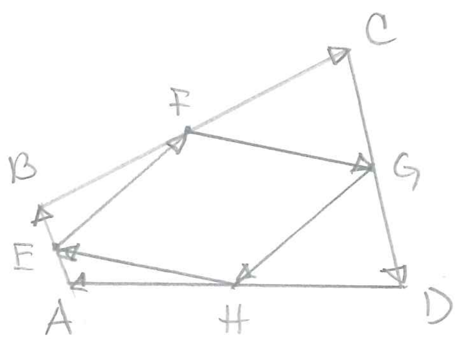
$$\sqrt{4a^2} = 1 \Rightarrow |2a| = 1$$

$$\Rightarrow a = \pm\frac{1}{2} \Rightarrow b = \pm\frac{\sqrt{3}}{2}$$

The unit vectors parallel to the tangent line at $\left(\frac{\pi}{6}, 1\right)$ are

$$\pm\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$$

10.



$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DA} = \vec{0}$ (1)

$\frac{1}{2}\vec{AB} + \frac{1}{2}\vec{BC} = \vec{EF}$ (2)

$\frac{1}{2}\vec{CD} + \frac{1}{2}\vec{DA} = \vec{GH}$ (3)

(2)+(3): $\frac{1}{2}\left(\vec{AB} + \vec{BC} + \vec{CD} + \vec{DA}\right) = \vec{EF} + \vec{GH}$

But since $\vec{AB} + \vec{BC} + \vec{CD} + \vec{DA} = \vec{0}$, we have that $\vec{EF} = -\vec{GH}$; which means that the line segment EF is equal in length and parallel to the segment GH. A similar argument is used to show $\vec{FG} = -\vec{HE}$.