Homework # 10

Math - 243 - Section 50

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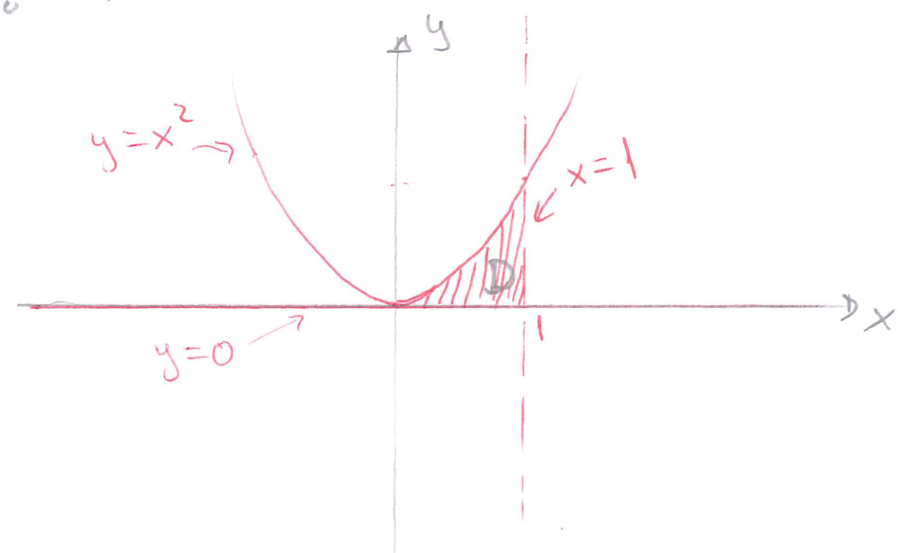
1. Let us first find the area of the region of integration. A sketch is below



$$A(D)=\iint_D dA = \int_0^1\int_0^{x^2} dy\,dx = \int_0^1 y\Big|_0^{x^2} dx = \int_0^1 x^2\,dx$$

$$= \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

Now, $$\iint_D f(x,y)\,dA = \iint_D x\sin y\,dA = \int_0^1\int_0^{x^2} x\sin y\,dy\,dx$$

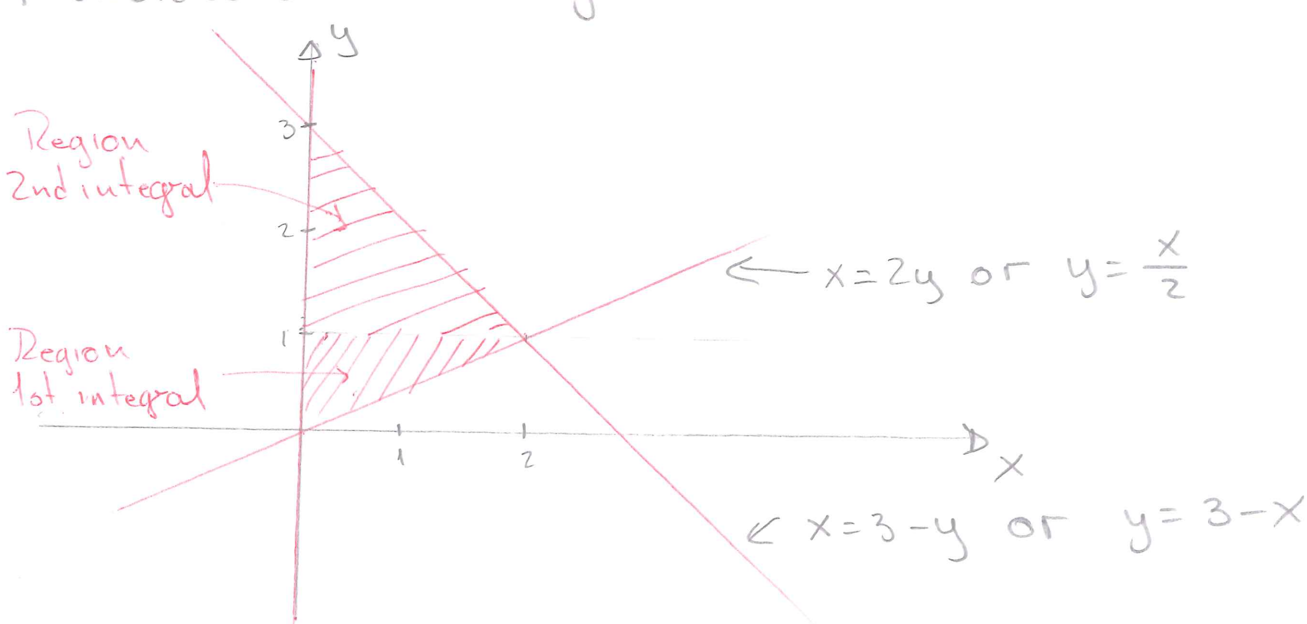
$$= \int_0^1 -x\cos y\Big|_0^{x^2} dx = \int_0^1 (-x\cos x^2 + x)\,dx$$

$$= \int_0^1 (x - x\cos x^2)\,dx = \frac{x^2}{2} - \frac{1}{2}\sin x^2 \Big|_0^1$$

$$= \frac{1}{2} - \frac{\sin 1}{2} = \frac{1}{2}\left(1 - \sin(1)\right)$$

$$\therefore \text{Average value} = \frac{\frac{1}{2}(1-\sin(1))}{\frac{1}{3}} = \boxed{\frac{3(1-\sin(1))}{2}}$$

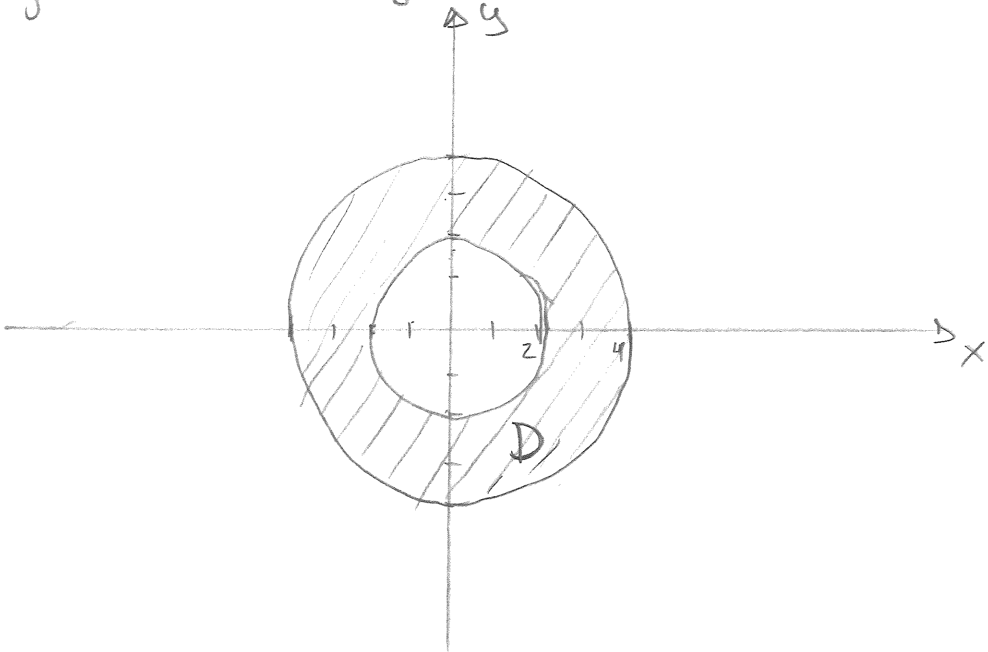
2. A sketch of the region D is shown below:



So,

$$\int_0^1\int_0^{2y} f(x,y)\,dx\,dy + \int_1^3\int_0^{3-y} f(x,y)\,dx\,dy = \underline{\int_0^2\int_{\frac{x}{2}}^{3-x} f(x,y)\,dy\,dx}$$

3. The region of integration is shown below:



Let us denote by V the volume we are trying to find. By symmetry then

$$\frac{V}{2} = \iint_D \sqrt{16-x^2-y^2}\, dA$$

Transforming into polar coordinates:

$$\frac{V}{2} = \int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\; r\, dr\, d\theta \qquad \begin{array}{l} u = 16-r^2 \\ du = -2r\,dr \end{array}$$

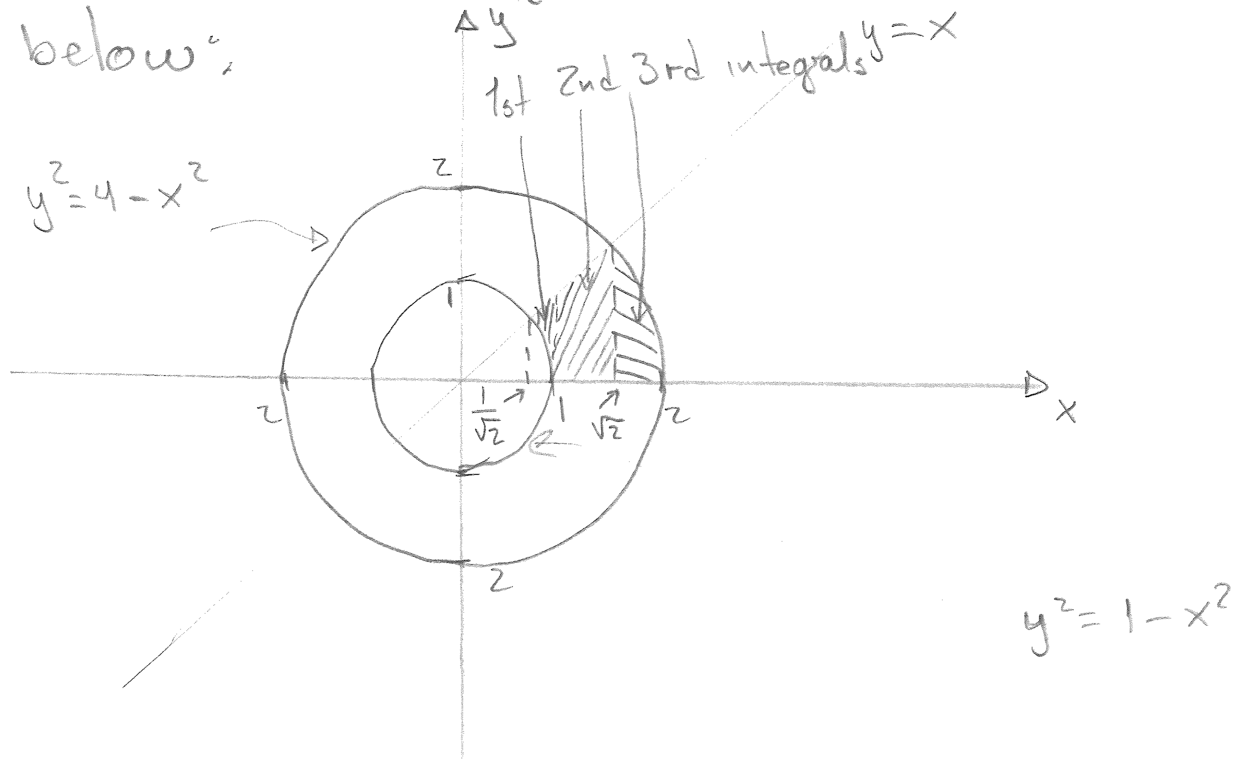
$$V = 2\int_0^{2\pi}\int_{12}^{0} -\frac{1}{2}\sqrt{u}\, du\, d\theta$$

$$= \int_0^{2\pi}\int_0^{12} \sqrt{u}\, du\, d\theta = \int_0^{2\pi} \frac{2}{3} u^{3/2}\Big|_0^{12} d\theta$$

$$V = \frac{2}{3}\int_0^{2\pi} \left(8\sqrt{27}\right) d\theta = \frac{16}{3}\sqrt{27}\,(2\pi)$$

$$= \frac{32\pi}{3}\sqrt{27} = \underline{32\pi\sqrt{3}}$$

4. A sketch of the region of integration is shown below:



$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx = \int_{0}^{\pi/4}\int_{1}^{2} r^2\cos\theta\sin\theta\, r\,dr\,d\theta$$

$$\int_{0}^{\pi/4}\int_{1}^{2} r^3\cos\theta\sin\theta\,dr\,d\theta = \int_{0}^{\pi/4} \frac{r^4}{4}\cos\theta\sin\theta\Big|_1^2\,d\theta$$

$$= \int_0^{\pi/4} \left(4 - \frac{1}{4}\right) \cos\theta \sin\theta \, d\theta = \frac{15}{4} \int_0^{\pi/4} \cos\theta \sin\theta \, d\theta$$

Using
$u = \sin\theta$
$du = \cos\theta \, d\theta$

$$= \frac{15}{4} \left(\frac{\sin^2\theta}{2} \right) \Bigg|_0^{\pi/4} = \frac{15}{4} \left(\frac{\left(\frac{\sqrt{2}}{2}\right)^2}{2} \right) = \frac{15}{4} \left(\frac{\frac{2}{4}}{2} \right)$$

$$= \frac{15}{4} \left(\frac{1}{4} \right) = \frac{15}{16}$$

5. $z = f(x,y) = y^2 - x^2 \Rightarrow \begin{array}{l} f_x = -2x = -2r\cos\theta \\ f_y = 2y = 2r\sin\theta \end{array}$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4r^2 + 1}$$

So

$$S = \int_0^{2\pi} \int_1^2 \sqrt{4r^2+1} \; r \, dr \, d\theta \qquad \begin{array}{l} u = 4r^2 + 1 \\ du = 8r\,dr \end{array}$$

$$= \frac{1}{8} \int_0^{2\pi} \int_5^{17} \sqrt{u} \; du \, d\theta = \frac{1}{8} \int_0^{2\pi} \frac{2}{3} u^{3/2} \Big|_5^{17} d\theta$$

$$= \frac{1}{8}\left(\frac{2}{3}\right)\int_0^{2\pi}\left(17\sqrt{17}-5\sqrt{5}\right)d\theta$$

$$= \frac{1}{12}\left(17\sqrt{17}-5\sqrt{5}\right)(2\pi) = \frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$

6.

$$\int_0^1\int_x^{2x}\int_0^y 2xyz\,dz\,dy\,dx = \int_0^1\int_x^{2x} xyz^2\Big|_0^y\,dy\,dx$$

$$= \int_0^1\int_x^{2x} xy^3\,dy\,dx = \int_0^1 x\frac{y^4}{4}\Big|_x^{2x}\,dx = \frac{1}{4}\int_0^1\left(16x^5 - x^5\right)dx$$

$$= \frac{1}{4}\int_0^1 15x^5\,dx = \frac{15}{4}\left(\frac{x^6}{6}\right)\Big|_0^1 = \frac{15}{24} = \frac{5}{8}$$

7.

$$\int_0^{\sqrt{\pi}}\int_0^x\int_0^{xz} x^2\sin y\,dy\,dz\,dx = \int_0^{\sqrt{\pi}}\int_0^x -x^2\cos y\Big|_0^{xz}\,dz\,dx$$

$$= \int_0^{\sqrt{\pi}}\int_0^x\left(-x^2\cos(xz) + x^2\right)dz\,dx$$

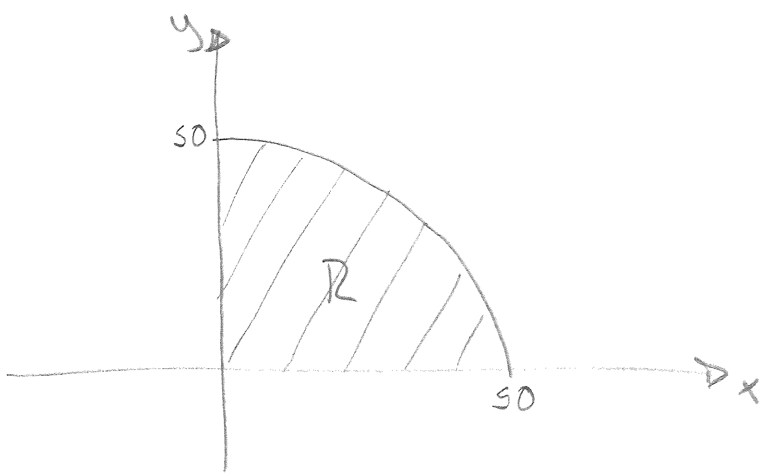
$$= \int_0^{\sqrt{\pi}} \int_0^{x} \left(x^2 - x^2 \cos(xz)\right) \, dz\, dx$$

$$= \int_0^{\sqrt{\pi}} x^2 z - \frac{x^2}{x} \sin(xz) \Big|_0^{x} \, dx$$

$$= \int_0^{\sqrt{\pi}} \left[x^3 - x \sin(x^2)\right] dx = \frac{x^4}{4} + \frac{1}{2}\cos(x^2) \Big|_0^{\sqrt{\pi}}$$

$$= \left[\frac{\pi^2}{4} + \frac{1}{2}(-1)\right] - \left[0 + \frac{1}{2}\right] = \boxed{\frac{\pi^2}{4} - 1}$$

8. A sketch of R:



a)

$$V = \iint_R \left(20 + \frac{xy}{100}\right) dA - \iint_R \left(\frac{x+y}{5}\right) dA$$

$$V = \iint\limits_R \left(20 + \frac{xy}{100} - \frac{(x+y)}{5}\right) dA$$

If $x = r\cos\theta$ and $y = r\sin\theta$

$$V = \int_0^{\pi/2}\int_0^{50} \left(20 + \frac{r^2 \sin\theta\cos\theta}{100} - \frac{r\cos\theta}{5} - \frac{r\sin\theta}{5}\right) r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{20r^2}{2} + \frac{r^4 \sin\theta\cos\theta}{400} - \frac{r^3\cos\theta}{15} - \frac{r^3\sin\theta}{15} \Bigg|_0^{60} d\theta$$

$$= \int_0^{\pi/2} \left[10(2500) + 15625\sin\theta\cos\theta - 8333.33\cos\theta - 8333.33\sin\theta\right] d\theta$$

$$= 25000\theta + \frac{15625}{2}\sin^2\theta - 8333.33\sin\theta + 8333.33\cos\theta \Bigg]_0^{\pi/2}$$

$$= \left(12500\pi + \frac{15625}{2} - 8333.33\right) - \left(8333.33\right)$$

$$\approx 30415.74 \text{ ft}^3$$

b) $f(x,y) = 20 + \frac{xy}{100}$

$f_x = \frac{y}{100}$; $f_y = \frac{x}{100}$; if $x = r\cos\theta$, $y = r\sin\theta$

$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{r^2}{(100)^2} + 1}$

$\therefore$

$S = \int_0^{\pi/2}\int_0^{50} \sqrt{\frac{r^2}{(100)^2} + 1} \; r \, dr \, d\theta$

$u = \frac{r^2}{(100)^2} + 1$

$du = \frac{2r}{(100)^2} dr$

$= \frac{(100)^2}{2} \int_0^{\pi/2} \int_1^{\left(\frac{50}{100}\right)^2 + 1} \sqrt{u} \, du \, d\theta$

$= \frac{(100)^2}{2} \int_0^{\pi/2} \int_1^{5/4} \sqrt{u} \, du \, d\theta = \frac{(100)^2}{2} \int_0^{\pi/2} \frac{2}{3} u^{3/2} \Big|_1^{5/4} d\theta$

$= \frac{(100)^2}{2} \left(\frac{2}{3}\right)\left(\left(\frac{5}{4}\right)^{3/2} - 1\right) \frac{\pi}{2}$

$\approx 2081.53 \text{ ft}^2$

9.

$$I=\int_0^1\int_0^{3-a-y^2}\int_a^{4-x-y^2} dz\,dx\,dy = \int_0^1\int_0^{3-a-y^2}(4-x-y^2-a)\,dx\,dy =$$

$$\int_0^1\left[4x-\frac{x^2}{2}-y^2x-ax\right]_0^{3-a-y^2}dy =$$

$$\int_0^1\underbrace{\left[4(3-a-y^2)-\frac{(3-a-y^2)^2}{2}-y^2(3-a-y^2)-a(3-a-y^2)\right]}_{A}dy$$

Now

$$(3-a-y^2)^2=9-6a-6y^2+2ay^2+a^2+y^4$$

$$A=\frac{15}{2}-4a-4y^2+ay^2+\frac{a^2}{2}+\frac{y^4}{2}$$

So

$$I=\int_0^1 A\,dy=\frac{15}{2}y-4ay-\frac{4}{3}y^3+\frac{a}{3}y^3+\frac{a^2}{2}y+\frac{y^5}{10}\Bigg|_0^1$$

$$=\frac{15}{2}-4a-\frac{4}{3}+\frac{a}{3}+\frac{a^2}{2}+\frac{1}{10}=\frac{14}{15}\quad \text{*Given}$$

$$\Rightarrow\quad 3a^2-22a+32=0$$

Using the quadratic formula:

$$a = 2 \quad \text{or} \quad a = \frac{16}{3}$$

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10. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \xrightarrow[\text{coordinates}]{\text{polar}} \int_0^{2\pi}\int_0^{a} e^{-r^2}\,r\,dr\,d\theta =$$

$$= -\frac{1}{2}\int_0^{2\pi} e^{-r^2}\Big|_0^a\,d\theta = -\frac{1}{2}\int_0^{2\pi}\left(e^{-a^2}-1\right)d\theta$$

$$= -\frac{1}{2}\left[2e^{-a^2}\pi - 2\pi\right] = \pi - e^{-a^2}\pi$$

Now

$$\lim_{a\to\infty}\left(\pi - e^{-a^2}\pi\right) = \boxed{\pi}$$