Homework #11

Math 243- Section 50

Dr. Marco A. Montes de Oca

1. $V(E) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1-x^2-y^2} dz\, dy\, dx =$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\, dy\, dx =$$

$$\int_{0}^{2\pi} \int_{0}^{1} (1-r^2)\, r\, dr\, d\theta = \int_{0}^{2\pi} \int_{0}^{1} (r-r^3)\, dr\, d\theta =$$

$$\int_{0}^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_{0}^{1} d\theta = \frac{1}{4}\int_{0}^{2\pi} d\theta = \frac{2\pi}{4} = \frac{\pi}{2}$$

Then $f_{ave} = \frac{1}{\frac{\pi}{2}} \iiint_E (x^2 z + y^2 z)\, dV = \frac{2}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1-x^2-y^2} (x^2+y^2)z\, dz\, dy\, dx$

$$= \frac{2}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)\left(\frac{1}{2}(1-x^2-y^2)^2\right) dy\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 r^2(1-r^2)^2\, r\, dr\, d\theta = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 (r^3 - 2r^5 - r^7)\, dr\, d\theta =$$

$$\frac{1}{\pi}\int_0^{2\pi} \frac{r^4}{4} - \frac{r^6}{3} + \frac{r^8}{8}\Big|_0^1 \, d\theta = \frac{1}{24\pi}\int_0^{2\pi} d\theta = \frac{2\pi}{24\pi} = \boxed{\frac{1}{12}}$$

2. The region of integration is the region above the cone $z = \sqrt{x^2+y^2}$, or $z = r$, and below the plane $z=2$. Also, $-2 \leq y \leq 2$ and $-\sqrt{4-y^2} \leq x \leq \sqrt{4-y^2}$ which is a circle of radius 2 in the xy-plane, centered at $(0,0)$.

So

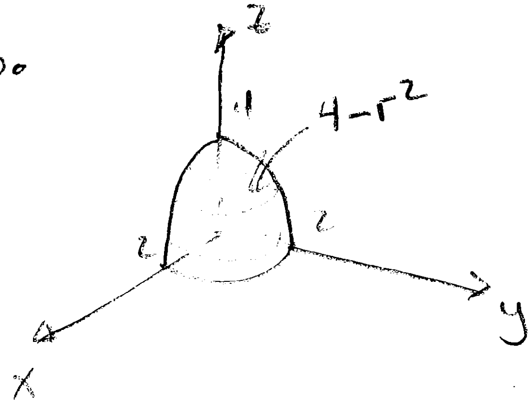
$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\, dz\, dx\, dy =$$

$$\int_0^{2\pi}\int_0^2\int_r^2 (r\cos\theta)\, z\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^2\int_r^2 r^2\cos\theta\, z\, dz\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 r^2\cos\theta\,(4-r^2)\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\cos\theta\left(\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 d\theta$$

$$= \frac{1}{2}\left(\frac{32}{3} - \frac{32}{5}\right)\sin\theta\Big|_0^{2\pi \to 0} = 0$$

3.



$x+y+z = r\cos\theta + r\sin\theta + z$

$$\int_0^{\pi/2}\int_0^2\int_0^{4-r^2} (r(\cos\theta+\sin\theta)+z)\, r \, dz\,dr\,d\theta =$$

$$\int_0^{\pi/2}\int_0^2\int_0^{4-r^2} (r^2(\cos\theta+\sin\theta)+rz)\, dz\,dr\,d\theta =$$

$$\int_0^{\pi/2}\int_0^2 \left[(4-r^2)r^2(\cos\theta+\sin\theta) + \frac{r(4-r^2)^2}{2}\right] dr\,d\theta =$$

$$\int_0^{\pi/2}\int_0^2 (4r^2-r^4)(\cos\theta+\sin\theta)\,dr\,d\theta + \frac{1}{2}\int_0^{\pi/2}\int_0^2 (16r-8r^3+r^5)\,dr\,d\theta =$$

$$\left.\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 \int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta + \frac{1}{2}\left(8r^2-2r^4+\frac{r^6}{6}\right)_0^2 \int_0^{\pi/2} d\theta =$$

$$\frac{64}{15}\left(\sin\theta-\cos\theta\right)_0^{\pi/2} + \frac{1}{2}\left(32-32+\frac{32}{3}\right)\frac{\pi}{2} =$$

$$\frac{64}{15}\left((1-0)-(0-1)\right) + \frac{8\pi}{3} =$$

$$\frac{128}{15}+\frac{8\pi}{3}$$

4. The region of integration is the region above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=2$ in the 1st octant. So the integral is equivalent to

$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\; d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} \rho^4\sin^3\phi\cos\theta\sin\theta\; d\rho\, d\theta\, d\phi =$$

$$\frac{\rho^5}{5}\Big]_0^{\sqrt{2}} \int_0^{\pi/4}\int_0^{\pi/2} \sin^3\phi\cos\theta\sin\theta\; d\theta\, d\phi =$$

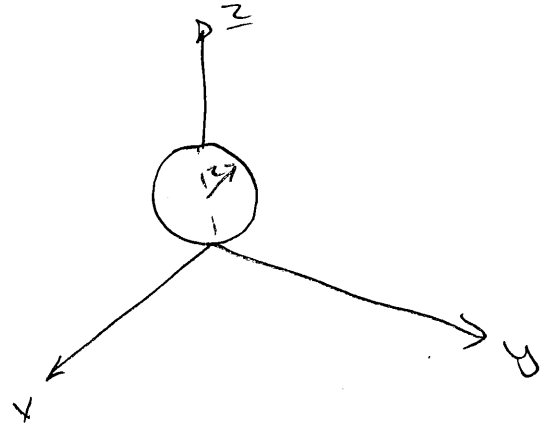
$$\frac{4\sqrt{2}}{5}\int_0^{\pi/4}\int_0^{\pi/2} (1-\cos^2\phi)\sin\phi\cos\theta\sin\theta\; d\theta\, d\phi =$$

$$\frac{4\sqrt{2}}{5}\left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{\pi/2}\int_0^{\pi/4}(\sin\phi-\cos^2\phi\sin\phi)\,d\phi =$$

$$\frac{2\sqrt{2}}{5}\left(\frac{1}{3}\cos^3\phi-\cos\phi\right)\Big]_0^{\pi/4} = \left(\frac{\sqrt{2}}{12}-\frac{\sqrt{2}}{2}-\left(\frac{1}{3}-1\right)\right)\frac{2\sqrt{2}}{5}$$

$$= \boxed{\frac{4\sqrt{2}-5}{15}}$$

5. The region of integration is a ball centered at $(0,0,2)$ with radius 2.



The equation of this ball is $\rho = 4\cos\phi$ and the integral becomes:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\cos\phi} \rho^3 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta =$$

$$\int_0^{2\pi}\int_0^{\pi/2} \left.\frac{\rho^6}{6}\right]_0^{4\cos\phi} \sin\phi\, d\phi\, d\theta = \frac{4096}{6}\int_0^{2\pi}\int_0^{\pi/2}\cos^6\phi\,\sin\phi\,d\phi\,d\theta$$

$$= \frac{4096\pi}{3}\int_0^{\pi/2}\cos^6\phi\,\sin\phi\, d\phi = \frac{4096\pi}{3}\left(\frac{1}{7}\right) = \boxed{\frac{4096\pi}{21}}$$

6. The given integral is equal to

$$\lim_{a\to\infty} \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho e^{-\rho^2}\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

If $u = \rho^2$
$dv = \rho e^{-\rho^2} d\rho$

then

$$= \lim_{a\to\infty} \int_0^{2\pi}\int_0^{\pi} \left[\left[\rho^2\left(-\frac{1}{2}\right)e^{-\rho^2}\right]_0^a - \int_0^a 2\rho\left(-\frac{1}{2}\right)e^{-\rho^2}\, d\rho \right] d\phi\, d\theta$$

$$= \lim_{a\to\infty} \left(-\frac{1}{2}a^2 e^{-a^2} - \frac{1}{2}e^{-a^2} + \frac{1}{2}\right) \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi$$

$$\lim_{a\to\infty} 4\pi\left(\underbrace{\cancel{\frac{1}{2}a^2 e^{-a^2}}}_{\to 0} - \underbrace{\cancel{\frac{1}{2}e^{-a^2}}}_{\to 0} + \frac{1}{2}\right) = 4\pi\left(\frac{1}{2}\right) = \boxed{2\pi}$$

7. $f(x,y) = \sqrt{x^2+y^2}$

$$\nabla f(x,y) = \left\langle \frac{2x}{2\sqrt{x^2+y^2}}, \frac{2y}{2\sqrt{x^2+y^2}} \right\rangle$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

∇f is not defined at $(0,0)$ but elsewhere all vectors have length 1 and point away from the origin.

8. $\vec{r}(t) = \langle 4t, 3+3t \rangle \quad 0 \le t \le 1,$

$\vec{r}\,'(t) = \langle 4, 3 \rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{16+9} = \sqrt{25} = 5$

So

$$\int_C x \sin y \, ds = \int_0^1 4t \sin(3+3t)(5)\, dt =$$

$$I = 20 \int_0^1 t \sin(3+3t)\, dt \Rightarrow \begin{matrix} u = t & dv = \sin(3+3t)\, dt \\ du = dt & v = -\frac{1}{3}\cos(3+3t) \end{matrix}$$

$$I = 20 \left[-\frac{1}{3} t \cos(3+3t) + \frac{1}{9} \sin(3+3t) \right]_0^1$$

$$= 20 \left[-\frac{1}{3}\cos 6 + \frac{1}{9} \sin 6 + 0 - \frac{1}{9} \sin 3 \right]$$

$$= \frac{20}{9} \left[\sin 6 - 3\cos 6 - \sin 3 \right]$$

9. $x = 2\sin t,\ y = t,\ z = -2\cos t, \quad 0 \le t \le \pi$

So

$$\int_C xyz\, ds = \int_0^{\pi} (2\sin t)(t)(-2\cos t) \sqrt{(2\cos t)^2 + (1)^2 + (2\sin t)^2}\, dt =$$

$$-2\sqrt{5}\int_0^{\pi} t\sin 2t\,dt \Rightarrow \left[\begin{array}{ll} u=t & dv=\sin(2t)\,dt \\ du=dt & v=-\frac{1}{2}\cos(2t)\end{array}\right.$$

$$=-2\sqrt{5}\left[-\frac{1}{2}t\cos 2t+\frac{1}{4}\sin 2t\right]_0^{\pi}$$

$$=-2\sqrt{5}\left(-\frac{\pi}{2}-0\right)=\boxed{\sqrt{5}\,\pi}$$

10. $A=\int_C 4+0.01\,(x^2-y^2)\,ds$

C: $\vec{r}(t)=\langle 10\cos t,\ 10\sin t\rangle$

$0\le t\le 2\pi$

$$\vec{r}\,'(t)=\langle -10\sin t,\ 10\cos t\rangle$$

$$\|\vec{r}\,'(t)\|=\sqrt{(-10\sin t)^2+(10\cos t)^2}=\sqrt{100(\sin^2 t+\cos^2 t)}$$

$$=10$$

$$A=\int_0^{2\pi}\left[4+0.001\,(100\cos^2 t-100\sin^2 t)\right](10)\,dt$$

$$=40(2\pi)+\int_0^{2\pi}\cos^2 t\,dt-\int_0^{2\pi}\sin^2 t\,dt$$

$$=80\pi+0-0=80\pi$$ Area of one side of the fence

So, Area of both sides: $160\pi\approx 502.65\ m^2$

So, we need ≈ 5.02 L of paint