Homework #12

Math 243 - Section 50

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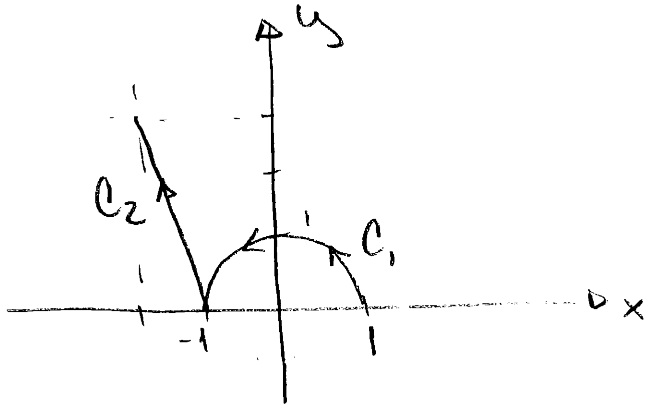
1.



$C = C_1 + C_2$

C_1: $x = \cos t$
$dx = -\sin t\, dt$

$y = \sin t$
$dy = \cos t\, dt$

$0 \le t \le \pi$

C_2: $x = -1 - t$, $dx = -dt$
$y = 3t$,
$dy = 3\,dt$

$0 \le t \le 1$

So

$$\int_C \sin x\, dx + \cos y\, dy = \int_{C_1} \sin x\, dx + \cos y\, dy \; +$$

$$\int_{C_2} \sin x\, dx + \cos y\, dy = I$$

$$I = \int_0^{\pi} \sin(\cos t)(-\sin t\, dt) + \cos(\sin t)\cos t\, dt \; +$$
$$\int_0^{1} \sin(-1-t)(-dt) + \cos(3t)(3)dt$$

$$I = \Big[-\cos(\cos t) + \sin(\sin t)\Big]_0^{\pi} + \Big[-\cos(-1-t) + \sin(3t)\Big]_0^1$$

$$= -\cos(\cos\pi) + \sin(\sin\pi) + \cos(\cos 0) - \sin(\sin 0)$$

$$- \cos(-2) + \sin(3) + \cos(-1) - \sin(0)$$

$$= -\cos(-1) + \sin 0 + \cos(1) - \sin 0 - \cos(-2)$$

$$+ \sin 3 + \cos(-1)$$

$$= -\cos 1 + \cos 1 - \cos 2 + \sin 3 + \cos 1$$

$$= \cos 1 - \cos 2 + \sin 3$$

2. $\vec{F}(\vec{r}(t)) = \langle t^2 + t^3, t^3 - t^2, (t^2)^2 \rangle = \langle t^2 + t^3, t^3 - t^2, t^4 \rangle$

$\vec{r}\,'(t) = \langle 2t, 3t^2, 2t \rangle$

So

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^1 (2t^3 + 2t^4 + 3t^5 - 3t^4 + 2t^5)\,dt$$

$$= \int_0^1 (5t^5 - t^4 + 2t^3)\,dt = \frac{5}{6}t^6 - \frac{t^5}{5} + \frac{t^4}{2}\Big]_0^1$$

$$= \frac{5}{6} - \frac{1}{5} + \frac{1}{2} = \frac{17}{15}$$

3.
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} \langle \cos t, \sin t, -t\rangle \cdot \langle 1, \cos t, -\sin t\rangle \, dt$$
$$= \int_0^{\pi} (\cos t + \sin t \cos t + t \sin t)\, dt$$
$$= \sin t + \frac{1}{2}\sin^2 t + (\sin t - t\cos t)\Big|_0^{\pi}$$
$$= \pi$$

4. $x = x$, $y = x^2$, $-1 \le x \le 2$
$$W = \int_{-1}^{2} \langle x \sin x^2, x^2\rangle \cdot \langle 1, 2x\rangle \, dx = \int_{-1}^{2} (x\sin x^2 + 2x^3)\, dx$$
$$= \left[-\frac{1}{2}\cos x^2 + \frac{1}{2}x^4\right]_{-1}^{2} = \frac{1}{2}(15 + \cos 1 - \cos 4) \approx 8.096$$

5. $\frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y$

$\frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$

$\neq \therefore \vec{F}$ is not conservative.

6. $\dfrac{\partial}{\partial y}\left(\dfrac{y^2}{1+x^2}\right) = \dfrac{2y}{1+x^2}$

$\dfrac{\partial}{\partial x}(2y \arctan x) = \dfrac{2y}{1+x^2}$

$=$ $\therefore \vec{F}$ is conservative.

IF $f_x = \dfrac{y^2}{1+x^2} \Rightarrow f = \int f_x \, dx = y^2 \arctan x + C(y)$

$\Rightarrow f_y = 2y \arctan x + C'(y) \Rightarrow$ $C'(y) = 0$

By comparing with $\vec{F}$

$\therefore C(y) = K$

So

$f(x,y) = y^2 \arctan x + K$

7. $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xz + y^2 & 2xy & x^2 + 3z^2 \end{vmatrix} = (0-0)\hat{\imath} - (2x - 2x)\hat{\jmath} + (2y - 2y)\hat{k} = \vec{0}$

$\therefore \vec{F}$ is conservative and we can use the FTC4LIs.

$f_x = 2xz + y^2 \Rightarrow f = \int f_x \, dx = x^2 z + xy^2 + c(y,z)$

$f_y = 2xy + c_y(y,z) \xRightarrow[\text{with } \vec{F}]{\text{Comparing}} c_y(y,z) = 0$

or $c(y,z) = w(z)$. Thus $f(x,y,z) = x^2 z + xy^2 + w(z)$

So, $f_z = x^2 + w'(z) \xRightarrow[\text{with } \vec{F}]{\text{Comparing}} w'(z) = 3z^2$

and $w(z) = z^3 + K$. We can conclud then:

$\underline{f(x,y,z) = x^2 z + xy^2 + z^3 + K}$

Using the FTC4LIs:

$\int_C \vec{F} \cdot d\vec{r} = f(1,2,1) - f(0,1,-1) = 6 - (-1) = \underline{7}$

8. $\vec{F} = \langle \tan y, \; x \sec^2 y \rangle$

$\frac{\partial}{\partial y} \tan y = \sec^2 y$

$\frac{\partial}{\partial x}(x \sec^2 y) = \sec^2 y$

$= \Rightarrow \vec{F}$ is conservative.

$f(x,y) = x\tan y + K$ so

$$\int_C \tan y\,dx + x\sec^2 y\,dy = f\left(2,\tfrac{\pi}{4}\right) - f(1,0) = 2\tan\tfrac{\pi}{4} - \tan 0$$

$$= 2$$

9.-

a) According to the FTC4LIs:

$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ where C starts at $t=a$ and ends at $t=b$. So, if

$\int_{C_1} \nabla f \cdot d\vec{r} = 0$, then $f(\vec{r}(b)) = f(\vec{r}(a))$. Choosing

$\vec{r}(b) = \langle 0,0\rangle \Rightarrow f(0,0) = 0$

$\vec{r}(a) = \langle \pi,0\rangle \Rightarrow f(\pi,0) = 0$

So one possible curve C_1 is

$\vec{r}(t) = \langle \pi - t, 0\rangle$, $0 \le t \le \pi$

b) Similarly, if

$$f(\vec{r}(b)) - f(\vec{r}(a)) = 1$$

Let's choose $\vec{r}(a) = \langle 0, 0 \rangle$ so that $f(\vec{r}(a)) = 0$.

Then $f(\vec{r}(b)) = 1$. This happens when $x - 2y = \frac{\pi}{4}$. Let $x = \pi$, then $y = \frac{3\pi}{8}$

So C_2 can be described by:

$$\boxed{\vec{r}(t) = \langle t, \tfrac{3}{8}t \rangle \quad 0 \le t \le \pi}$$

10.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln x & \ln xy & \ln xyz \end{vmatrix} = \left(\frac{xz}{xyz} - 0\right)\hat{i} - \left(\frac{yz}{xyz} - 0\right)\hat{j} + \left(\frac{y}{xy} - 0\right)\hat{k}$$

$$= \left\langle \frac{1}{y}, -\frac{1}{x}, \frac{1}{x} \right\rangle$$