Homework #2
Math 243 - Section 50
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1. Written document.

2. First, we need to create vectors that represent the edges of the triangle.

$$\vec{AB} = \langle -2-0, 4-0, 1-1 \rangle = \langle -2, 4, 0 \rangle$$
$$\vec{AC} = \langle 4-0, 2-0, 1-1 \rangle = \langle 4, 2, 0 \rangle$$
$$\vec{BC} = \langle 4-(-2), 2-4, 1-1 \rangle = \langle 6, -2, 0 \rangle$$

The triangle can then be represented by the equation

$$\vec{AB} + \vec{BC} = \vec{AC}$$

To determine whether the triangle is an acute, obtuse or right triangle, we need to determine the angles betwen the vectors defined above.

The angle between $\vec{AB}$ and $\vec{AC}$ is given by

$$\vec{AB} \cdot \vec{AC} = \|\vec{AB}\| \, \|\vec{AC}\| \cos\theta, \Rightarrow \cos\theta = \frac{\vec{AB} \cdot \vec{AC}}{\|\vec{AB}\| \, \|\vec{AC}\|}$$

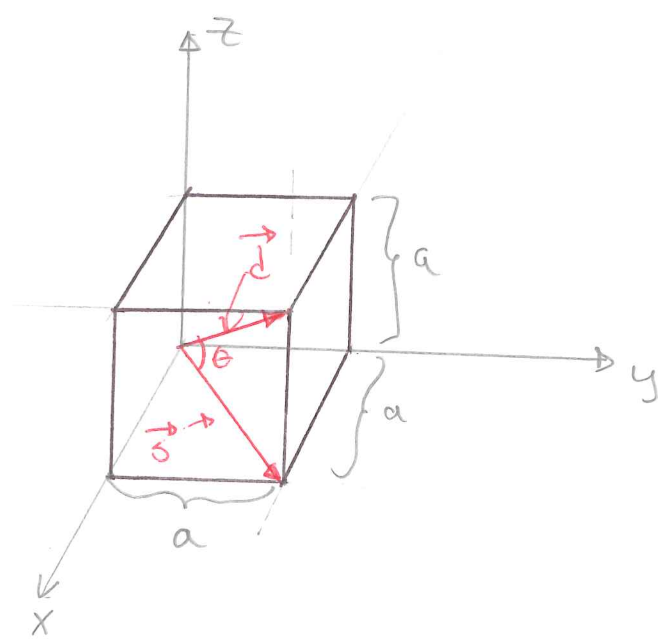
Using the data we have:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \langle -2,4,0 \rangle \cdot \langle 4,2,0 \rangle = -8+8=0$$

$\therefore \quad \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

Since one of the angles of the triangle ABC is equal to $\frac{\pi}{2}$, the triangle given is a right triangle.

3. The figure below shows the situation:



We want to find the angle θ between vectors $\vec{d}$ and $\vec{s}$.

According to the figure:

$$\vec{d} = \langle a, a, a \rangle \quad \text{and} \quad \vec{b} = \langle a, a, 0 \rangle$$

So

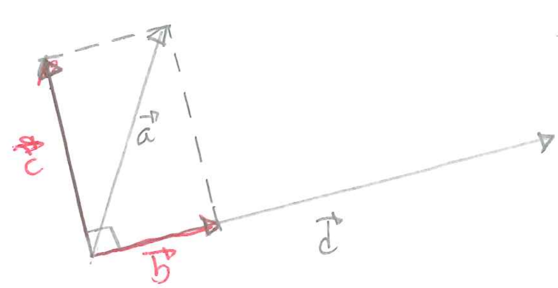
$$\cos\theta = \frac{\vec{d} \cdot \vec{b}}{\|\vec{d}\| \, \|\vec{b}\|} = \frac{a^2 + a^2 + a(0)}{\left(\sqrt{3a^2}\right)\left(\sqrt{2a^2}\right)} = \frac{2a^2}{a^2\sqrt{3}\sqrt{2}} = \sqrt{\frac{2}{3}}$$

$$\therefore \ \underline{\theta = \arccos\left(\sqrt{\frac{2}{3}}\right) = 0.615 \text{ rad} \approx 35.26^\circ}$$

4. The displacement vector is $\vec{d} = \langle 6-1, 1-2, -1-8 \rangle$
$= \langle 5, -1, -9 \rangle$.

So
$$W = \vec{F} \cdot \vec{d} = \langle 3, -2, 1 \rangle \cdot \langle 5, -1, -9 \rangle = 15 + 2 - 9 = \underline{8 \text{ J}}$$

5. The situation is depicted below



We want to find $\vec{b}$ and $\vec{c}$ such that $\vec{a} = \vec{b} + \vec{c}$.

From the figure, it is apparent that

$\|\vec{b}\| = \text{comp}_{\vec{d}}\,\vec{a} = \dfrac{\vec{a}\cdot\vec{d}}{\|\vec{d}\|}$ and since $\vec{b}$ is parallel to $\vec{d}$ we can write $\vec{c}$ as

$$\vec{b} = \left(\text{comp}_{\vec{d}}\,\vec{a}\right)\hat{d} = \frac{\vec{a}\cdot\vec{d}}{\|\vec{d}\|}\,\frac{\vec{d}}{\|\vec{d}\|}$$

$$= \left(\frac{\vec{a}\cdot\vec{d}}{\|\vec{d}\|^2}\right)\vec{d} = \left(\frac{\vec{a}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\right)\vec{d}$$

So,

$$\frac{\langle 3,2,-6\rangle\cdot\langle 2,-4,1\rangle}{\langle 2,-4,1\rangle\cdot\langle 2,-4,1\rangle} = \frac{6-8-6}{4+16+1} = \frac{-8}{21}$$

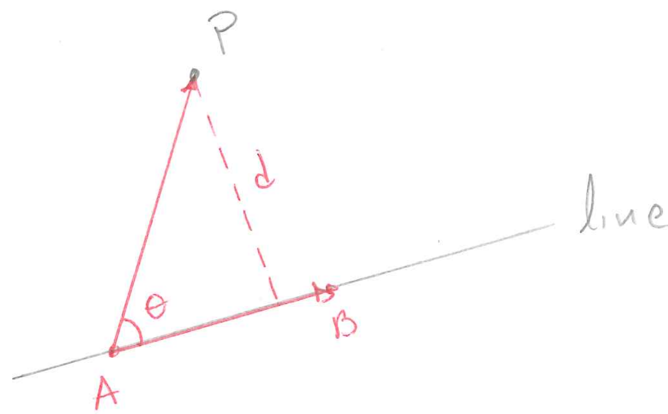
$$\therefore\ \vec{b} = -\frac{8}{21}\langle 2,-4,1\rangle$$

Now, also from the figure:

$$\vec{a} - \vec{b} = \vec{c}\ \therefore\ \vec{c} = \langle 3,2,-6\rangle - \left(-\frac{8}{21}\right)\langle 2,-4,1\rangle$$

$$= \left\langle 3 + \frac{16}{21},\ 2 - \frac{32}{21},\ -6 + \frac{8}{21}\right\rangle$$

6. The situation is shown below



Note that $d = \|\vec{AP}\| \sin\theta$. Since $\|\vec{AB} \times \vec{AP}\| = \|\vec{AB}\|\,\|\vec{AP}\| \sin\theta$ we can write

$$d = \|\vec{AP}\| \sin\theta = \frac{\|\vec{AB} \times \vec{AP}\|}{\|\vec{AB}\|} \quad (1)$$

To use equation (1), we need two points on the line to generate vectors $\vec{AB}$ and $\vec{AP}$.

In our case, the line is $y = 4x + 1$.

@ $x = 0$, $y = 4(0) + 1 = 1$, $A(0, 1)$

@ $x = 1$, $y = 4(1) + 1 = 5$, $B(1, 5)$

$\therefore \vec{AB} = \langle 1-0, 5-1, 0-0 \rangle = \langle 1, 4, 0 \rangle$

$\vec{AP} = \langle 1-0, 2-1, 0-0 \rangle = \langle 1, 1, 0 \rangle$

$$\vec{AB} \times \vec{AP} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0\hat{i} - 0\hat{j} + (1-4)\hat{k} = \langle 0, 0, -3 \rangle$$

$$\|\vec{AB} \times \vec{AP}\| = 3$$

Now

$$\|\vec{AB}\| = \sqrt{1+16} = \sqrt{17}$$

$$\therefore \quad \boxed{d = \frac{3}{\sqrt{17}}}$$

7. If $\vec{a} = \alpha \vec{b}$, that is, if $\vec{a}$ is parallel to $\vec{b}$, then

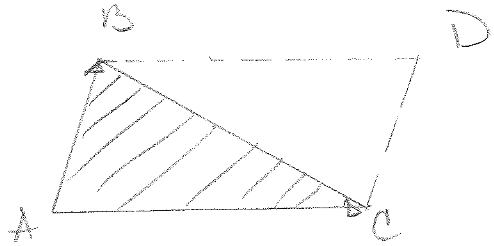
$$\vec{a} \times \vec{b} = \alpha \vec{b} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha b_1 & \alpha b_2 & \alpha b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (\alpha b_2 b_3 - b_2(\alpha b_3))\hat{i} - (\alpha b_1 b_3 - b_1(\alpha b_3))\hat{j} + (\alpha b_1 b_2 - b_1(\alpha b_2))\hat{k}$$

$$= \langle 0, 0, 0 \rangle = \vec{0}$$

8. First create vectors that represent the edges of the triangle.

$$\vec{AB} = \langle 0-2, 1-(-3), 2-4 \rangle = \langle -2, 4, -2 \rangle$$
$$\vec{AC} = \langle -1-0, 2-1, 0-2 \rangle = \langle -1, 1, -2 \rangle$$

Geometrically



The area of the triangle ABC is half the area of the parallelogram ABDC. So

$$\text{Area triangle} = \frac{1}{2}\|\vec{AC} \times \vec{AB}\|$$

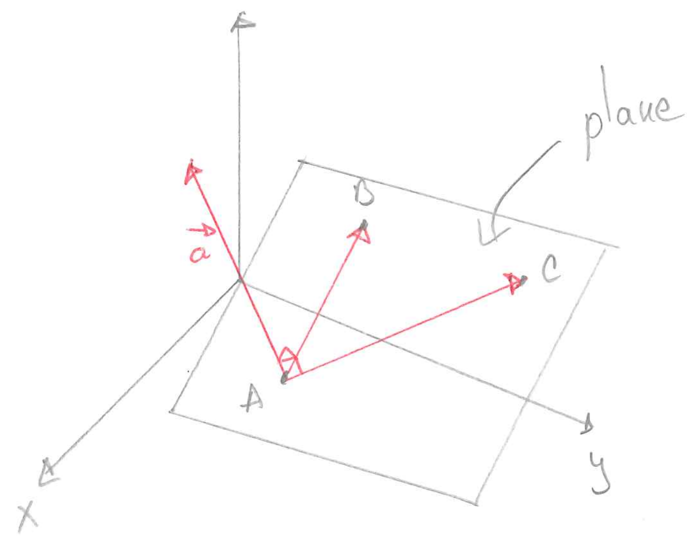
$$\vec{AC} \times \vec{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ -2 & 4 & -2 \end{vmatrix} = (-2-(-8))\hat{i} - (2-4)\hat{j} + (-4-(-2))\hat{k}$$

$$= 6\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\|\vec{AC} \times \vec{AB}\| = \sqrt{36+4+4} = \sqrt{44} = 2\sqrt{11}$$

$$\therefore \text{Area triangle} = \frac{1}{2}(2\sqrt{11}) = \underline{\sqrt{11}}$$

9. The situation is depicted below



We want to find $\hat{a} = \dfrac{\vec{a}}{\|\vec{a}\|}$, which is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

So, $\overrightarrow{AB} = \langle 1-0, 0-1, 1-1 \rangle = \langle 1, -1, 0 \rangle$

$\overrightarrow{AC} = \langle 1-0, 1-1, 0-1 \rangle = \langle 1, 0, -1 \rangle$

$$\vec{a} = \overrightarrow{AC} \times \overrightarrow{AB} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = (1-0)\hat{\imath} - (-1-0)\hat{\jmath} + (0-(-1))\hat{k}$$

$$= \hat{\imath} + \hat{\jmath} + \hat{k}$$

$\|\vec{a}\| = \sqrt{3}$

$\therefore \hat{a} = \dfrac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$