Homework #3

Math 243 - Section 50

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1. Let $\vec{r}_1 = \langle 1-3t, t, -1-3t \rangle$ represent the given line. Its direction vector is $\vec{v}_1 = \langle -3, 1, -3 \rangle$ because we can write

$$\vec{r}_1 = \langle 1, 0, -1 \rangle + t \underbrace{\langle -3, 1, -3 \rangle}_{\text{direction vector}}.$$

The line we are looking for passes through $(1, 2, 10)$ and is parallel to $\vec{r}_1$, therefore we can say

$$\vec{r} = \langle 1, 2, 10 \rangle + s \underbrace{\langle -3, 1, -3 \rangle}_{\substack{\text{same direction} \\ \text{vector as } \vec{r}_1}}$$

$$= \langle 1-3s, 2+s, 10-3s \rangle$$

2. If the line we are looking for is perpendicular to the plane $4x-y-z=3$, then it must be parallel to the plane's normal vector $\vec{n}$.

By inspection of the equation $4x-y-z=3$, we can say that $\vec{n}=\langle 4,-1,-1\rangle$ (It's the vector whose components are equal to the coefficients of x, y, and z). Thus, our line is given by

$$\vec{r}=\underbrace{\langle -1,1,1\rangle}_{\text{given point}}+t\langle 4,-1,-1\rangle$$

$$=\langle -1+4t,1-t,1-t\rangle$$

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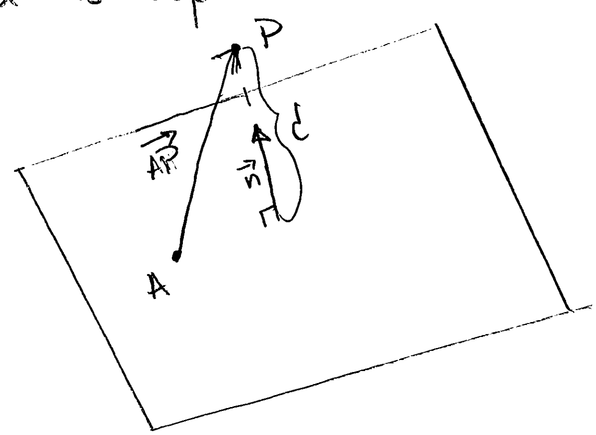
3. The situation is depicted below



From the drawing, it is apparent that $d = \left|\text{comp}_{\vec{n}} \vec{AP}\right|$ (the projection of $\vec{AP}$ onto $\vec{n}$).

Thus,

$$d = \left|\text{comp}_{\vec{n}} \vec{AP}\right| = \left|\frac{\vec{AP} \cdot \vec{n}}{\|\vec{n}\|}\right|$$

Now, $\vec{n} = \langle 1,1,1\rangle$ (from the equation of the plane). $\vec{AP}$ is a vector from a point on the plane to point P. So, if $x = y = 0$, then according to the equation of the plane $z = 1$. A point on the plane is thus $(0,0,1)$, and $\vec{AP} = \langle 1-0, 0-0, -1-1\rangle$ $= \langle 1, 0, -2\rangle$.

$\vec{AP} \cdot \vec{n} = \langle 1,0,-2\rangle \cdot \langle 1,1,1\rangle = 1 + 0 - 2 = -1$

and $\|\vec{n}\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

So, $d = \left|\frac{-1}{\sqrt{3}}\right| = \boxed{\frac{1}{\sqrt{3}}}$

4. Based on the discussion we had in class, the distance between two non-parallel lines is given by

$$d = \left| \frac{\vec{PP} \cdot (\vec{v}_1 \times \vec{v}_2)}{\|\vec{v}_1 \times \vec{v}_2\|} \right| , \quad \text{where}$$

$\vec{PP}$ is a vector from one point on one line to a point on the other line, $\vec{v}_1$ and $\vec{v}_2$ are the direction vectors of the lines involved.

In this case,

$$\vec{v}_1 = \langle 2, -1, 3 \rangle \quad \text{and} \quad \vec{v}_2 = \langle 4, -2, 5 \rangle$$

So

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix} = (-5+6)\hat{\imath} - (10-12)\hat{\jmath} + (-4+4)\hat{k}$$

$$= \hat{\imath} + 2\hat{\jmath} = \langle 1, 2, 0 \rangle$$

$$\|\vec{v}_1 \times \vec{v}_2\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

If we take the reference points of each line, then

$$\vec{PP} = \langle 1-3, 3-4, 4-1 \rangle = \langle -2, -1, 3 \rangle$$

$$\vec{PP} \cdot (\vec{v}_1 \times \vec{v}_2) = \langle -2,-1,3\rangle \cdot \langle 1,2,0\rangle$$

$$= -2-2+0 = -4$$

$$\therefore \quad d = \left|\frac{-4}{\sqrt{5}}\right| = \boxed{\frac{4}{\sqrt{5}}}$$

5. Let us number the equations of the planes we are given:

(1) $x - z = 1$ (2) $y + 2z = 3$ (3) $x + y - 2z = 1$

We want to find an equation of a plane that passes through the line of intersection of (1) and (2), but that is perpendicular to (3).

Let us first find the line of intersection of (1) and (2). We need to find a point on that line and its direction vector.

A point on the line must satisfy simultaneously equations (1) and (2). Since we have three unknowns and two equations, we can propose the value of one of the variables and use the equations to find the values of the other variables.

So, let us suppose that $x=0$. From ①, if $x=0$ $z=-1$. Now, if $x=0$ and $z=-1$, then from ② we see that $y=5$. Therefore, the point $(0,5,-1)$ is on the line.

To find the intersection line's direction vector, we must observe that such a vector is perpendicular to the normal vectors of the intersecting planes. So by crossing those normal vectors, we can find the intersection line's direction vector.

From ①, $\vec{n}_1 = \langle 1,0,-1 \rangle$ and from ② $\vec{n}_2 = \langle 0,1,2 \rangle$.
Thus, let us denote by $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} =$

$= \hat{i} - 2\hat{j} + \hat{k} = \langle 1,-2,1 \rangle$

We can then say that the vector equation of the intersection line of ① and ② is:

$$\vec{r} = \langle 0,5,-1 \rangle + t\langle 1,-2,1 \rangle$$

Now, the normal of the plane we are looking for must be perpendicular to the normal of (3) or $\vec{n}_3 = \langle 1, 1, -2 \rangle$ and to any vector parallel to the plane, including the direction vector of the intersection line.

So, we can compute the plane's normal vector as

$$\vec{n} = \vec{v} \times \vec{n}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 3\hat{i} + 3\hat{j} + 3\hat{k} = \langle 3, 3, 3 \rangle$$

The scalar equation of the plane is given by

$$3(x-0) + 3(y-5) + 3(z+1) = 0$$

coordinates of point on intersection line, and therefore on the plane.

Simplifying:

$$\underline{x + y + z = 4}$$

6. From points $A(1,3,2)$, $B(3,-1,6)$, $C(5,2,0)$. and $D(3,6,-4)$ we can generate vectors $\vec{AB}$, $\vec{AC}$, and $\vec{AD}$. If the points lie in the same plane, then these vectors also lie in the same plane.

If that is the case, then the volume of the parallelepiped spanned by these vectors must be equal to zero. So let's see whether

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = 0$$

$$\vec{AB} = \langle 3-1, -1-3, 6-2 \rangle = \langle 2, -4, 4 \rangle$$

$$\vec{AC} = \langle 5-1, 2-3, 0-2 \rangle = \langle 4, -1, -2 \rangle$$

$$\vec{AD} = \langle 3-1, 6-3, -4-2 \rangle = \langle 2, 3, -6 \rangle$$

$$\vec{AC} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = (6-(-6))\hat{i} - (-24-(-4))\hat{j} + (12-(-2))\hat{k}$$
$$= \langle 12, 20, 14 \rangle$$

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \langle 2,-4,4 \rangle \cdot \langle 12,20,14 \rangle = 24 - 80 + 56 = \underline{0}$$

$\therefore$ The points A, B, C, and D lie in the same plane.

An equation of this plane can be found if $\vec{n} = \overrightarrow{AC} \times \overrightarrow{AD} = \langle 12, 20, 14 \rangle$. Using A as a reference point, we have

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

$$12x - 12 + 20y - 60 + 14z - 28 = 0$$

$$12x + 20y + 14z - 100 = 0$$

or

$$\underline{6x + 10y + 7z - 50 = 0}$$

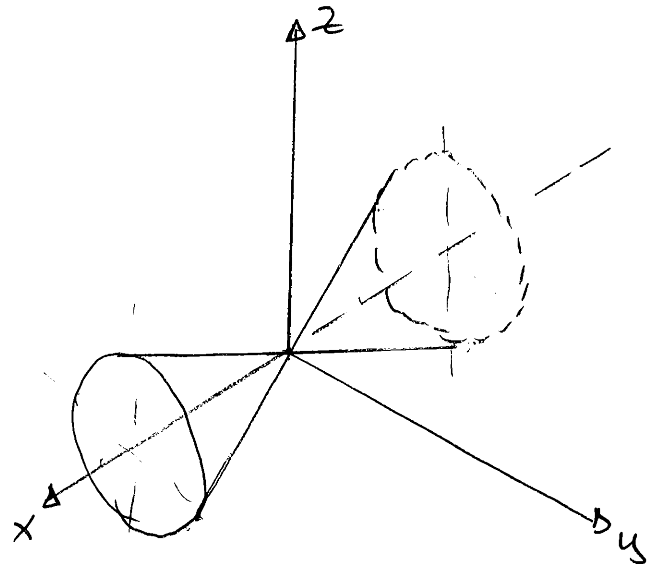
7. $x^2 = 2y^2 + 3z^2$

$$= \frac{y^2}{\frac{1}{2}} + \frac{z^2}{\frac{1}{3}} = \frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{z^2}{\left(\frac{1}{\sqrt{3}}\right)^2} \quad \text{or}$$

multiplying everything by $\frac{1}{6}$:

$$\frac{x^2}{6} = \frac{y^2}{3} + \frac{z^2}{2}$$

In any case, this equation represents a cone (See Table 1 Ch 12.6) aligned along the x-axis. A rough sketch of the cone is:

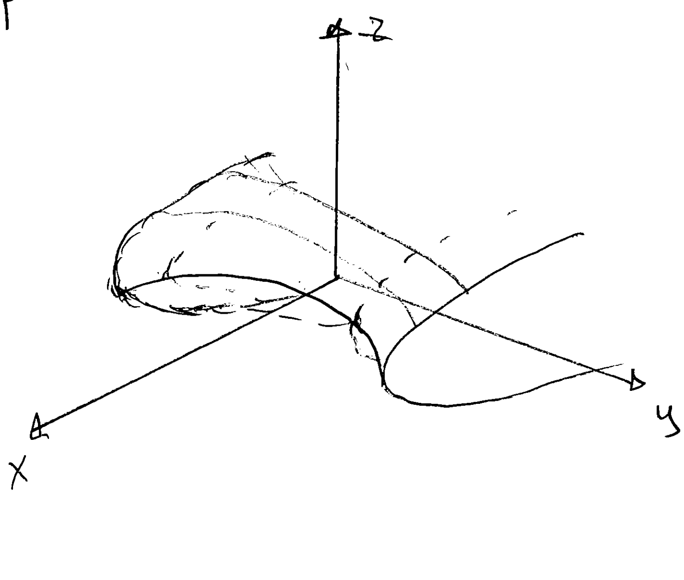


8. $4x - y^2 + 4z^2 = 0$

$$4x = y^2 - 4z^2$$

$$x = \frac{y^2}{2} - z^2$$

Based on Table 1, Ch 12.6, this equation represents a hyperbolic paraboloid with the "saddle" aligned with the y-axis and oriented toward the positive x-axis.



9. $4y^2+z^2-x-16y-4z+20=0$

Rearranging terms:

$$4y^2-16y+z^2-4z=x-20$$

Completing squares:

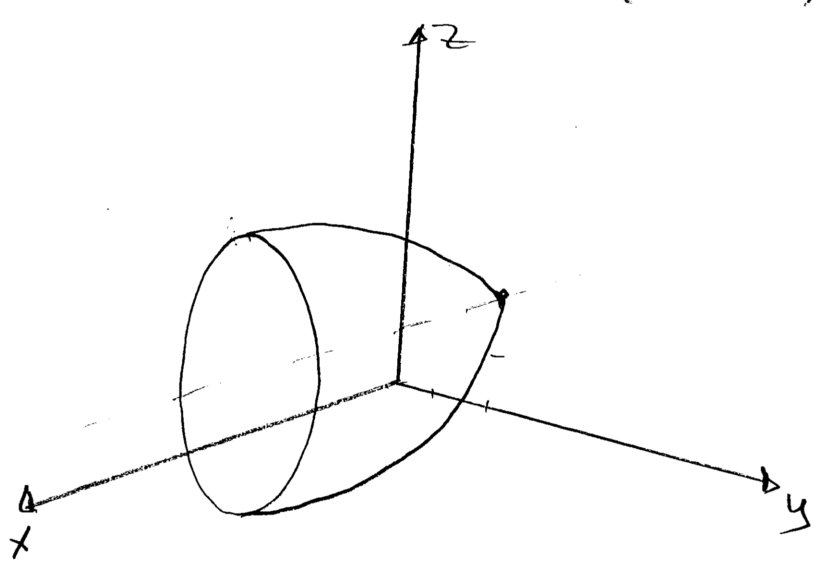
$$4(y^2-4y)+z^2-4z=x-20$$

$$4(y^2-4y+4)+(z^2-4z+4)=x-20+16+4$$

$$4(y-2)^2+(z-2)^2=x$$

$$(y-2)^2+\frac{(z-2)^2}{4}=\frac{x}{4}$$

Based on table 1, Ch 12.6 this is an elliptic paraboloid with vertex at (0,2,2).



10. $x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0$

Regrouping and completing squares:

$$(x^2 - 2x) - (y^2 - 2y) + (z^2 + 4z) = -2$$

$$(x-1)^2 - (y-1)^2 + (z+2)^2 = -2 + 1 - 1 + 4 = 2$$

$$\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2} + \frac{(z+2)^2}{2} = 1$$

This is a hyperboloid with vertex at $(1,1,-2)$ aligned with the y-axis.