Homework #4
Math-243 - Section 50
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1. If $\vec{r}(t) = \langle 3-t, 1-t^2, 1+\frac{1}{t^2} \rangle$ passes through $(2,0,2)$, then there must be a value t_c, for which

$$\vec{r}(t_c) = \langle 3-t_c, 1-t_c^2, 1+\frac{1}{t_c^2} \rangle = \langle 2, 0, 2 \rangle$$

$\therefore$

$3 - t_c = 2$ (1)

$1 - t_c^2 = 0$ (2)

$1 + \frac{1}{t_c^2} = 2$ (3)

From (1):

$t_c = 3 - 2 = 1$ (4)

(4) in (2):

$1 - (1)^2 = 0$ [Satisfied]

(4) in (3):

$1 + \frac{1}{(1)^2} = 2$ [Satisfied]

$\therefore$ $\vec{r}(t)$ passes through $(2,0,2)$ when $t = 1$.

For $(4,0,2)$:

$$3 - t_c = 4 \quad (1)$$
$$1 - t_c^2 = 0 \quad (2)$$
$$1 + \frac{1}{t_c^2} = 2 \quad (3)$$

From (1)

$$t_c = 3 - 4 = -1 \quad (4)$$

(4) in (2)

$$1 - (-1)^2 = 0 \quad [\text{Satisfied}]$$

(4) in (3)

$$1 + \frac{1}{(-1)^2} = 2 \quad [\text{Satisfied}]$$

$\therefore$ $\vec{r}(t)$ passes through $(4,0,2)$ when $t = -1$.

For $(3,1,2)$:

$$3 - t_c = 3 \quad (1)$$
$$1 - t_c^2 = 0 \quad (2)$$
$$1 + \frac{1}{t_c^2} = 2 \quad (3)$$

From (1):

$$t_c = 3 - 3 = 0 \quad (4)$$
$$1 - (0)^2 = 1 \neq 0 \quad [\text{Contradiction}]$$

$\therefore$ $\vec{r}(t)$ cannot possibly pass through $(3,1,2)$.

2. If the particles collide, then

$$\vec{r}_1(t^*) = \vec{r}_2(t^*)$$

So, $\vec{r}_1(t^*) = \langle t^*, (t^*)^2, (t^*)^3 \rangle = \langle 1+2t^*, 1+6t^*, 1+14t^* \rangle$

$t^* = 1 + 2t^*$ (1)

$(t^*)^2 = 1 + 6t^*$ (2)

$(t^*)^3 = 1 + 14t^*$ (3)

From (1):

$0 = 1 + 2t^* - t^*$

$0 = 1 + t^* \Rightarrow t^* = -1$ (4)

(4) in (2)

$(-1)^2 = 1 + (6)(-1)$

$1 = -5$ [Contradiction]

$\therefore$ the two objects <u>do not</u> collide.

Let us see whether the trajectories intersect. In this case it is enoug that

$t = 1 + 2s$ (1)

$t^2 = 1 + 6s$ (2)

$t^3 = 1 + 14s$ (3)

for some t and s.

① in ②:

$$(1+2s)^2 = 1+6s$$

$$1+4s+4s^2 = 1+6s$$

$$4s^2 - 2s = 0$$

$$s(4s-2) = 0 \Rightarrow \begin{array}{l} s=0 \\ 4s-2=0 \Rightarrow s=\frac{1}{2} \end{array}$$

$\therefore$ $t=1$ or $t=2$

If $t=1$ & $s=0$ in ③:

$(1)^3 = 1+14(0) \Rightarrow$ [Satisfied]

If $t=2$ & $s=\frac{1}{2}$ in ③:

$(2)^3 = 1+14\left(\frac{1}{2}\right) =$

$8 = 1+7 = 8$ [Satisfied]

$\vec{r}_1(t)$ & $\vec{r}_2(s)$ intersect twice. Once at $(1,1,1)$ and once at $(2,4,8)$.

3. $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), \tan(t) \rangle$

$\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), \sec^2(t) \rangle$

$\vec{r}\,'(\pi/4) = \langle -\frac{\sqrt{2}}{2}(2), 2\frac{\sqrt{2}}{2}, 2 \rangle$

$= \langle -\sqrt{2}, \sqrt{2}, 2 \rangle$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \Rightarrow \vec{T}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{8}} \langle -\sqrt{2}, \sqrt{2}, 2 \rangle$

4. $\vec{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle$

$\vec{r}\,'(t) = \langle e^{2t}(2), e^{-2t}(-2), e^{2t} + 2te^{2t} \rangle$

$= \langle 2e^{2t}, -2e^{-2t}, (1+2t)e^{2t} \rangle$

$\vec{r}\,''(t) = \langle 4e^{2t}, 4e^{-2t}, 2e^{2t} + (1+2t)2e^{2t} \rangle$

$= \langle 4e^{2t}, 4e^{-2t}, 4e^{2t}(1+t) \rangle$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \Rightarrow \vec{T}(0) = \frac{\vec{r}\,'(0)}{\|\vec{r}\,'(0)\|}$

$\vec{r}\,'(0) = \langle 2, -2, 1 \rangle \; ; \; \|\vec{r}\,'(0)\| = \sqrt{9} = 3$

$\therefore \; \vec{T}(0) = \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$

$\vec{r}''(0) = \langle 4, 4, 4 \rangle$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2e^{2t} & -2e^{-2t} & (1+2t)e^{2t} \\ 4e^{2t} & 4e^{-2t} & 4(1+t)e^{2t} \end{vmatrix} =$$

$$= \left(-8(1+t)e^{0} - 4(1+2t)e^{0}\right)\hat{i}$$
$$-\left(8(1+t)e^{4t} - 4(1+2t)e^{4t}\right)\hat{j}$$
$$+\left(8e^{0} + 8e^{0}\right)\hat{k} = \left(-8-8t-4-8t\right)\hat{i}$$
$$-\left(e^{4t}(8+8t-4-8t)\right)\hat{j}$$
$$+16\hat{k}$$
$$= -(12+16t)\hat{i}$$
$$-4e^{4t}\hat{j}$$
$$+16\hat{k}$$
$$= \langle -(12+16t), -4e^{4t}, 16 \rangle$$

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5. A line can be fully determined by a point and a direction vector. In this case, the point is given, $(0,1,0)$ and for direction vector we can use $\vec{r}'(t_c)$ so that $\vec{r}(t_c) = \langle 0,1,0 \rangle$.

So, given $\vec{r}(t) = \langle t, e^{-t}, 2t - t^2 \rangle$

$\vec{r}(t_c) = \langle t_c, e^{-t_c}, 2t_c - t_c^2 \rangle = \langle 0, 1, 0 \rangle$

$\Rightarrow t_c = 0.$

Now, $\vec{r}\,'(t) = \langle 1, -e^{-t}, 2 - 2t \rangle$

and

$\vec{r}\,'(t_c) = \langle 1, -1, 2 \rangle$

So, the line is given by

$\vec{w}(s) = \langle 0, 1, 0 \rangle + s\langle 1, -1, 2 \rangle$

$= \langle s, 1 - s, 2s \rangle$ or $x = s$, $y = 1 - s$, $z = 2s$

6. Proceeding in the same way as above:

$\vec{r}(t) = \langle \ln t, 2\sqrt{t}, t^2 \rangle$

$\vec{r}(t_c) = \langle \ln t_c, 2\sqrt{t_c}, t_c^2 \rangle = \langle 0, 2, 1 \rangle$

$\Rightarrow t_c = 1.$

$\vec{r}'(t) = \langle \frac{1}{t}, \frac{1}{\sqrt{t}}, 2t \rangle$

$\vec{r}'(t_c) = \vec{r}'(1) = \langle 1, 1, 2 \rangle$

The tangent line is thus

$$\vec{w}(s) = \langle 0, 2, 1 \rangle + s\langle 1, 1, 2 \rangle$$

$$= \langle s, 2+s, 1+2s \rangle \quad \text{or} \quad \begin{aligned} x &= s \\ y &= 2+s \\ z &= 1+2s \end{aligned}$$

7.

$$\int_0^1 \left(\frac{4}{1+t^2}\hat{j} + \frac{2t}{1+t^2}\hat{k} \right) dt$$

$$= \int_0^1 \frac{4\,dt}{1+t^2}\hat{j} + \int_0^1 \frac{2t}{1+t^2}dt\,\hat{k}$$

$$= 4\int_0^1 \frac{dt}{1+t^2}\hat{j} + \int_0^1 \frac{2t\,dt}{1+t^2}\hat{k}$$

$$= 4\arctan(t)\hat{j} + \ln|1+t^2|\hat{k}\Big|_0^1$$

$$= 4\left(\frac{\pi}{4}\right)\hat{j} + \ln(2)\hat{k} - \left(4(0)\hat{j} + \cancel{\ln(1)}^{\,0}\hat{k}\right) = \underline{\langle 0, \pi, \ln(2) \rangle}$$

8. $\vec{r}\,'(t) = \langle t, e^t, te^t \rangle$ & $\vec{r}(0) = \langle 1, 1, 1 \rangle$

$$\vec{r}(t) = \int \vec{r}\,'(t)\, dt$$

$$= \left\langle \int t\, dt, \int e^t\, dt, \int te^t\, dt \right\rangle$$

$$= \left\langle \frac{t^2}{2}, e^t, e^t(t-1) \right\rangle + \vec{C}$$

$$\vec{r}(0) = \langle 0, 1, -1 \rangle + \vec{C} = \langle 1, 1, 1 \rangle$$

$$\Rightarrow \vec{C} = \langle 1, 0, 2 \rangle$$

$$\therefore \vec{r}(t) = \left\langle \frac{t^2}{2} + 1, e^t, e^t(t-1) + 2 \right\rangle$$

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9. $$\frac{d}{dt}\left[\vec{u}(t) \cdot \left(\vec{v}(t) \times \vec{w}(t)\right)\right] = \vec{u}\,'(t) \cdot \left(\vec{v}(t) \times \vec{w}(t)\right) + \vec{u}(t) \cdot \left(\vec{v}(t) \times \vec{w}(t)\right)'$$

$$= \vec{u}\,'(t) \cdot \left(\vec{v}(t) \times \vec{w}(t)\right) + \vec{u}(t) \cdot \left(\vec{v}\,'(t) \times \vec{w}(t) + \vec{v} \times \vec{w}\,'(t)\right)$$

$$= \vec{u}\,'(t) \cdot \left(\vec{v}(t) \times \vec{w}(t)\right) + \vec{u}(t) \cdot \left(\vec{v}\,'(t) \times \vec{w}(t)\right) + \vec{u}(t) \cdot \left(\vec{v}(t) \times \vec{w}\,'(t)\right)$$

10. If $\vec{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$, then

$$\vec{r}\,'(t) = \langle 2t, 5, 2t - 16 \rangle$$

and speed is

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 25 + (2t-16)^2}$$

$$= \sqrt{4t^2 + 25 + 4t^2 - 64t + 256}$$

$$= \sqrt{8t^2 - 64t + 281}$$

the speed will be minimum when $\frac{d\,\|\vec{r}\,'(t)\|}{dt} = 0$,

so

$$\frac{d}{dt}\|\vec{r}\,'(t)\| = \frac{16t - 64}{2\sqrt{8t^2 - 64t + 281}} = \frac{8t - 32}{\sqrt{8t^2 - 64t + 281}}$$

which will be zero only when

$$8t - 32 = 0$$

or

$$t = 4$$

So, at $t = 4$, the particle will be moving with minimum speed. [It is not a maximum because if $t < 4$, the speed will be decreasing and if $t > 4$, the speed will be increasing.]