Homework #5
Math 243 — Section 50
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1. If the curve $\vec{r}(t) = \langle \sin t, \cos t, t \rangle$ intersects the sphere $x^2 + y^2 + z^2 = 5$, then

$$\underbrace{\sin^2 t + \cos^2 t}_{1} + t^2 = 5$$

$$1 + t^5 = 5$$

$$t^2 = 5 - 1 = 4 \Rightarrow t = \pm 2$$

$\therefore$ The points of intersection are

$$\vec{r}(-2) = \langle \sin(-2), \cos(-2), -2 \rangle : (-0.909, -0.416, -2)$$

$$\vec{r}(2) = \langle \sin(2), \cos(2), 2 \rangle : (0.909, -0.416, 2)$$

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2. The intersection of $z = 4x^2 + y^2$ and $y = x^2$ can be parametrized as follows:

$$x = t$$

$$y = x^2 = t^2$$

$$z = 4x^2 + y^2 = 4t^2 + (t^2)^2 = 4t^2 + t^4$$

Therefore, the curve of intersection is given by

$$\vec{r}(t) = \langle t, t^2, 4t^2 + t^4 \rangle$$

3. $\vec{r}(t) = \langle t, 1-t, 3+t^2 \rangle$, $\vec{p}(s) = \langle 3-s, s-2, s^2 \rangle$

If $\vec{r}(t)$ and $\vec{p}(s)$ intersect, then

$$t = 3 - s \quad (1)$$
$$1 - t = s - 2 \quad (2)$$
$$3 + t^2 = s^2 \quad (3)$$

(1) in (2)

$$1 - (3 - s) = s - 2$$
$$1 - 3 + s = s - 2$$
$$-2 + s = s - 2$$
$$1 = 1 \quad \leftarrow \text{provides no information}$$

(1) in (3)

$$3 + (3-s)^2 = s^2$$
$$3 + 9 - 6s + s^2 = s^2$$
$$12 = 6s \Rightarrow s = 2 \quad (4)$$

(4) in (1)

$$t = 3 - 2 = 1$$

The curves intersect when

$t = 1$ & $s = 2$

The intersection point is $\underline{(1, 0, 4)}$.

To find their angle of intersection, we need two vectors tangent to the curves. These vectors are

$\vec{r}'(t) = \langle 1, -1, 2t \rangle$ @ $t=1$ $\vec{r}'(1) = \langle 1, -1, 2 \rangle$

$\vec{p}'(s) = \langle -1, 1, 2s \rangle$ @ $s=2$ $\vec{p}'(2) = \langle -1, 1, 4 \rangle$

Since $\vec{r}'(1) \cdot \vec{p}'(2) = \|\vec{r}'(1)\| \, \|\vec{p}'(2)\| \cos\theta$

$$\cos\theta = \frac{-1-1+8}{\sqrt{6}\,\sqrt{18}} = \frac{6}{\sqrt{3}\cdot 6} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \underline{0.955 \text{ rad or } 54.73^\circ}$$

4. A circular trajectory of radius a is given by

$\vec{r}(t) = \langle a\cos(t), a\sin(t) \rangle$

The velocity of a particle along $\vec{r}(t)$ is $\vec{r}'(t)$,

so

$\vec{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$

The acceleration is given by $\vec{r}''(t)$, so

$$\vec{r}''(t) = \langle -a\cos(t), -a\sin(t) \rangle$$

If $\vec{r}'(t)$ is perpendicular to $\vec{r}''(t)$, then $\vec{r}'(t) \cdot \vec{r}''(t)$ should equal zero.

Verifying that, we have

$$\langle -a\sin(t), a\cos(t) \rangle \cdot \langle -a\cos(t), -a\sin(t) \rangle =$$

$$a^2\sin(t)\cos(t) - a^2\sin(t)\cos(t) \underline{= 0}$$

5. The length of $\vec{r}(t)$ from $t=0$ to $t=1$ is given by

$$s = \int_0^1 \|\vec{r}'(t)\| \, dt$$

$$\vec{r}'(t) = \langle 12, 8\left(\tfrac{3}{2}\right)t^{1/2}, 6t \rangle = \langle 12, 12t^{1/2}, 6t \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{144 + 144t + 36t^2} = \sqrt{(6t+12)^2} = |6t+12|$$

since in $0 \leq t \leq 1$, $6t+12 > 0$, $|6t+12| = 6t+12$.

Therefore

$$s = \int_0^1 (6t+12)\,dt = (3t^2+12t)\Big|_0^1 = 3+12 = \underline{15}$$

6. Since the curve of intersection is closed, it is convenient to use periodic functions to parametrize the vector function that will describe the intersection curve.

In this case, we need first to transform $4x^2+y^2=4$ into its standard form:

$$x^2 + \frac{y^2}{4} = 1$$

$$\left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \quad (1)$$

So, the idea is to pick x and y such that (1) is satisfied. This can be done if $x = \cos t$ and $y = 2\sin t$, because

$$\left(\frac{\cos t}{1}\right)^2 + \left(\frac{2\sin t}{2}\right)^2 = 1$$

Lastly, since $x+y+z=2$, $z=2-x-y$
$$=2-\cos t-2\sin t$$

Thus, the vector function we are looking for is

$$\vec{r}(t)=\langle \cos t, 2\sin t, 2-\cos t-2\sin t\rangle$$

Now

$$\vec{r}\,'(t)=\langle -\sin t, 2\cos t, \sin t-2\cos t\rangle$$

$$\|\vec{r}\,'(t)\|=\sqrt{\sin^2 t+4\cos^2 t+(\sin t-2\cos t)^2}$$

$$=\sqrt{\sin^2 t+4\cos^2 t+\sin^2 t-4\sin t\cos t+4\cos^2 t}$$

$$=\sqrt{2\sin^2 t+8\cos^2 t-4\sin t\cos t}$$

So [Integrating numerically]

$$s=\int_0^{2\pi}\sqrt{2\sin^2 t+8\cos^2 t-4\sin t\cos t}\,dt\approx 13.519$$

7. $\vec{r}(t) = \langle e^{2t}\cos 2t,\ 2,\ e^{2t}\sin 2t \rangle$

$$\vec{r}\,'(t) = \langle e^{2t}(2)\cos 2t + e^{2t}(-\sin 2t)(2),\ 0,\ e^{2t}(2)\sin 2t + e^{2t}(\cos 2t)(2) \rangle$$

$$= \langle 2e^{2t}\cos 2t - 2e^{2t}\sin 2t,\ 0,\ 2e^{2t}\sin 2t + 2e^{2t}\cos 2t \rangle$$

$$= 2e^{2t}\langle \cos 2t - \sin 2t,\ 0,\ \sin 2t + \cos 2t \rangle$$

$$\|\vec{r}\,'(t)\| = 2e^{2t}\sqrt{(\cos 2t - \sin 2t)^2 + (\sin 2t + \cos 2t)^2}$$

$$= 2e^{2t}\sqrt{\cos^2 2t - 2\sin(2t)\cos(2t) + \sin^2 2t \rightarrow + \sin^2 2t + 2\sin(2t)\cos(2t) + \cos^2 2t}$$

continues

$$= 2e^{2t}\sqrt{2} = 2\sqrt{2}\,e^{2t}$$

Now

$$s(t) = \int_0^t 2\sqrt{2}\,e^{2u}\,du = 2\sqrt{2}\int_0^t e^{2u}\,du = 2\sqrt{2}\left[\left(\tfrac{1}{2}\right)e^{2t} - \left(\tfrac{1}{2}\right)e^{0}\right]$$

$$= \sqrt{2}\left(e^{2t} - 1\right)$$

Therefore

$$s = \sqrt{2}\left(e^{2t} - 1\right)$$

So

$$\frac{s}{\sqrt{2}} + 1 = e^{2t}$$

$$\ln\left(\frac{s}{\sqrt{2}} + 1\right) = 2t \Rightarrow t = \frac{1}{2}\ln\left(\frac{s}{\sqrt{2}} + 1\right) \quad (1)$$

Substituting (1) in original vector function:

$$\vec{r}(t) = \Big\langle e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)} \cos\left(2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\right),\ 2,$$
$$e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)} \sin\left(2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\right)\Big\rangle$$

$$= \Big\langle \left(\frac{s}{\sqrt{2}} + 1\right) \cos\left(\ln\left(\frac{s}{\sqrt{6}} + 1\right)\right),\ 2,$$
$$\left(\frac{s}{\sqrt{2}} + 1\right) \sin\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right)\Big\rangle$$

8. $\vec{r}(t) = \langle t, t, (1+t^2) \rangle$

$\vec{r}\,'(t) = \langle 1, 1, 2t \rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{2+4t^2}$

$\vec{r}\,''(t) = \langle 0, 0, 2 \rangle$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2\hat{i} - 2\hat{j} = \langle 2, -2, 0 \rangle$$

$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \sqrt{8} = 2\sqrt{2}$

$$\therefore \quad K(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{2\sqrt{2}}{(2+4t^2)^{3/2}}$$

$$K(0) = \frac{2\sqrt{2}}{(2)^{3/2}} = 1$$

9. An equation of a parabola that passes through the origin is

$$y = Ax^2 \quad \text{or} \quad x = By^2$$

Let us take $x = By^2$ in this case.

This curve has parametric equations:

$$x = Bt^2$$
$$y = t$$
$$z = 0$$

So $\vec{r}(t) = \langle Bt^2, t, 0 \rangle$

$\vec{r}\,'(t) = \langle 2Bt, 1, 0 \rangle$

$\vec{r}\,''(t) = \langle 2B, 0, 0 \rangle$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2Bt & 1 & 0 \\ 2B & 0 & 0 \end{vmatrix} = -2B\hat{k}$$

$\Rightarrow \|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = |-2B| = 2|B|$

$\|\vec{r}\,'(t)\| = \sqrt{4B^2t^2 + 1}$

So

$$\kappa(t) = \frac{2|B|}{(4B^2t^2+1)^{3/2}}$$

at $t=0$, that is, at the origin we have

$\kappa(0) = 2|B| = 16 \leftarrow$ given

$\Rightarrow |B| = 8 \Rightarrow \underline{B = \pm 8}$

Thus the parabola with the given requrements is

$$\underline{x = 8y^2 \quad \text{or} \quad x = -8y^2}$$

10. $\vec{r}(t) = \langle \sinh t, \cosh t, t \rangle$

$\vec{r}\,'(t) = \langle \cosh t, \sinh t, 1 \rangle$

$\vec{r}\,''(t) = \langle \sinh t, \cosh t, 0 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{\cosh^2 t + \sinh^2 t + 1}$

Since $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x+e^{-x}}{2}\right)^2 + \left(\frac{e^x-e^{-x}}{2}\right)^2 =$$

$$\frac{1}{4}\left[(e^x+e^{-x})^2+(e^x-e^{-x})^2\right] =$$

$$\frac{1}{4}\left[e^{2x}+2e^xe^{-x}+e^{-2x}+e^{2x}-2e^xe^{-x}+e^{-2x}\right] =$$

$$\frac{1}{4}\left[2e^{2x}+2e^{-2x}\right] = \frac{e^{2x}+e^{-2x}}{2} = \underline{\cosh 2x}$$

Therefore

$$\|\vec{r}\,'(t)\| = \sqrt{\cosh 2t + 1}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cosh t & \sinh t & 1 \\ \sinh t & \cosh t & 0 \end{vmatrix}$$

$$= \langle -\cosh t, \sinh t, \cosh^2 t - \sinh^2 t \rangle$$

But

$$\cosh^2 t - \sinh^2 t = \left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2 =$$

$$\frac{1}{4}\left[(e^t+e^{-t})^2-(e^t-e^{-t})^2\right] = \frac{1}{4}\left[e^{2t}+2e^te^{-t}+e^{-2t} - (e^{2t}-2e^te^{-t}+e^{-2t})\right]$$

$$= \frac{4}{4} = 1$$

Thus

$$\vec{r}' \times \vec{r}'' = \langle -\cosh t, \sinh t, 1 \rangle$$

$$\|\vec{r}' \times \vec{r}''\| = \sqrt{\cosh^2 t + \sinh^2 t + 1}$$

$$= \sqrt{\cosh 2t + 1} \quad \left(\text{Check } \|\vec{r}'(t)\|\right)$$

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{\sqrt{\cosh 2t + 1}}{(\cosh 2t + 1)^{3/2}} = \frac{1}{\cosh 2t + 1}$$

at $(0,1,0)$ $t = 0$, therefore

$$\kappa(0) = \frac{1}{\cosh 0 + 1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$