Homework #6
Math 243 – Section 50
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1. A plane curve in the xy-plane can be represented by $\vec{r}(t) = \langle t, f(t), 0 \rangle$. The curvature of $\vec{r}(t)$ is given by $\kappa(t) = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$. So

$\vec{r}\,'(t) = \langle 1, f'(t), 0 \rangle$ and $\vec{r}\,''(t) = \langle 0, f''(t), 0 \rangle$.

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix} = f''(t)\,\hat{k}$$

$$\Rightarrow \|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = |f''(t)|$$

Furthermore, $\|\vec{r}\,'(t)\| = \left(1 + (f'(t))^2\right)^{1/2}$

So $$\kappa(t) = \frac{|f''(t)|}{\left(1 + (f''(t))^2\right)^{3/2}} \qquad (1)$$

At an inflection point $f''(t) = 0$. Therefore, by (1) $\kappa(t) = 0$ at an inflection point.

2. The ellipse $x^2+4y^2=4$ can be parametrized as $\vec{r}(\theta)=\langle 2\cos\theta, \sin\theta, 0\rangle$. This is because

$$x^2+4y^2=4 \Rightarrow \frac{x^2}{4}+y^2=1 \Rightarrow \left(\frac{x}{2}\right)^2+y^2=1$$

if $x=2\cos\theta$ and $y=\sin\theta$, then

$$\left(\frac{2\cos\theta}{2}\right)^2+(\sin\theta)^2=\cos^2\theta+\sin^2\theta=1.$$

Then, $\vec{r}'(\theta)=\langle -2\sin\theta, \cos\theta, 0\rangle$ and $\vec{r}''(\theta)=\langle -2\cos\theta, -\sin\theta, 0\rangle$

$$\vec{r}'(\theta)\times\vec{r}''(\theta)=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin\theta & \cos\theta & 0 \\ -2\cos\theta & -\sin\theta & 0 \end{vmatrix}=\left(2\sin^2\theta+2\cos^2\theta\right)\hat{k}$$
$$=2\hat{k}$$

$$\|\vec{r}'(\theta)\times\vec{r}''(\theta)\|=2$$

$$\|\vec{r}'(\theta)\|=\sqrt{(-2\sin\theta)^2+\cos^2\theta}=\sqrt{4\sin^2\theta+\cos^2\theta}$$
$$=\sqrt{3\sin^2\theta+1}$$

$$\kappa(\theta)=\frac{\|\vec{r}'(\theta)\times\vec{r}''(\theta)\|}{\|\vec{r}'(\theta)\|^3}=\frac{2}{(3\sin^2\theta+1)^{3/2}}$$

$\kappa(\theta)$ is maximum when $\sin\theta = 0$, which occurs when $\theta = 0$ and $\theta = \pi$. These points correspond to the endpoints of the major axis $(-2, 0)$ and $(2,0)$. The curvature at these points is $\kappa(\theta) = 2$.

$\kappa(\theta)$ is minimum when $\sin\theta = \pm 1$, which occurs when $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. These points are $(0,1)$ and $(0,-1)$. At these points, the curvature $\kappa(\theta) = \frac{2}{8} = \frac{1}{4}$.

3. The safety of the Batmobile depends on the normal component of the acceleration because it determines the maximum centrifugal force that the Batmobile's tires can resist.

If the normal component of the acceleration remains constant as the Batmobile moves, then this means that the Batmobile travels as fast as it can within the safety limits.

Therefore, if the Batmobile travels at 30 mph when $x=1$, the normal component of the acceleration can be found as follows:

$$a_N(t) = \kappa(t)\left(\frac{ds}{dt}\right)^2$$

If $y=\frac{x^3}{3}$ is transformed into $\vec{r}(t) = \langle t, \frac{t^3}{3}, 0\rangle$ then

$$a_N(t) = \kappa(t)\,\|\vec{r}\,'(t)\|^2$$

at $t=x=1$:

$$a_N(1) = \kappa(1)\,\|\vec{r}\,'(1)\|^2$$

at $t=x=\frac{3}{2}$:

$$a_N\left(\tfrac{3}{2}\right) = \kappa\left(\tfrac{3}{2}\right)\|\vec{r}\,'\left(\tfrac{3}{2}\right)\|^2$$

But since the normal component of the acceleration remains constant, then

$$a_N\left(\tfrac{3}{2}\right) = a_N(1)$$

Therefore

$$\kappa\left(\tfrac{3}{2}\right)\|\vec{r}\,'\left(\tfrac{3}{2}\right)\|^2 = \kappa(1)\,\|\vec{r}\,'(1)\|^2$$

$$\Rightarrow \|\vec{r}\,'\left(\tfrac{3}{2}\right)\| = \sqrt{\frac{\kappa(1)}{\kappa\left(\tfrac{3}{2}\right)}\|\vec{r}\,'(1)\|^2}$$

$$\|\vec{r}\,'(\tfrac{3}{2})\| = \|\vec{r}\,'(1)\| \sqrt{\frac{\kappa(1)}{\kappa(\frac{3}{2})}} \qquad (1)$$

Now, $\kappa(t)$ is given by

$$\kappa(t) = \frac{2t}{(1+t^4)^{3/2}}$$

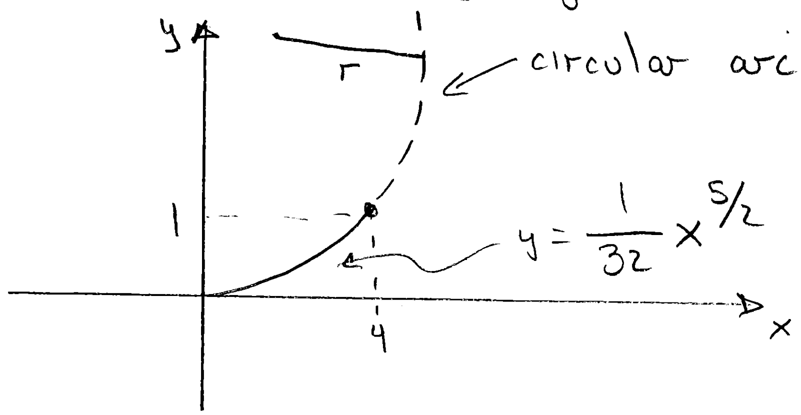
and $\kappa(1) = \dfrac{1}{\sqrt{2}}$; $\kappa(\tfrac{3}{2}) = \dfrac{192}{97\sqrt{97}}$

So

$$\|\vec{r}\,'(\tfrac{3}{2})\| = 30\sqrt{\frac{\frac{1}{\sqrt{2}}}{\frac{192}{97\sqrt{97}}}} = 30\sqrt{\frac{97\sqrt{97}}{192\sqrt{2}}}$$

$$\approx 56.27 \text{ mph}$$

4. Consider the following figure:



The radius of the circular arc, r, is the reciprocal of the circle's curvature (as we proved in class). Therefore $r = \frac{1}{\kappa}$ or $\kappa = \frac{1}{r}$.

Thus, by knowing the curvature at (4,1), we can find the arc's radius.

We are told that κ at (4,1) is equal to the curvature of $y = \frac{1}{32}x^{5/2}$ at (4,1).

Using the result of exercise 1, we know that

$$\kappa = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}$$

Therefore,

$$\kappa = \frac{\frac{15}{128}x^{1/2}}{\left(1 + \frac{25}{4096}x^3\right)^{3/2}}$$

at (4,1):

$$r = \frac{1}{\kappa} = \frac{\left(1 + \frac{25}{4096}(4)^3\right)^{3/2}}{\frac{15}{128}(4)^{1/2}} \approx 7$$

Both curves should have the same curvature so that the acceleration (in the normal direction) does not change abruptly when passing through the connection.

5. $f(x,y) = \int_x^y (2t-3)\,dt = 2\frac{t^2}{2} - 3t \Big]_x^y$

$= y^2 - 3y - (x^2 - 3x) = y^2 - x^2 - 3y + 3x$

So $f(0,4) = 4^2 - 3(4) = 16 - 12 = \underline{4}$

$f(1,4) = 4^2 - 1^2 - 3(4) + 3(1) = 16 - 1 - 12 + 3 = \underline{6}$

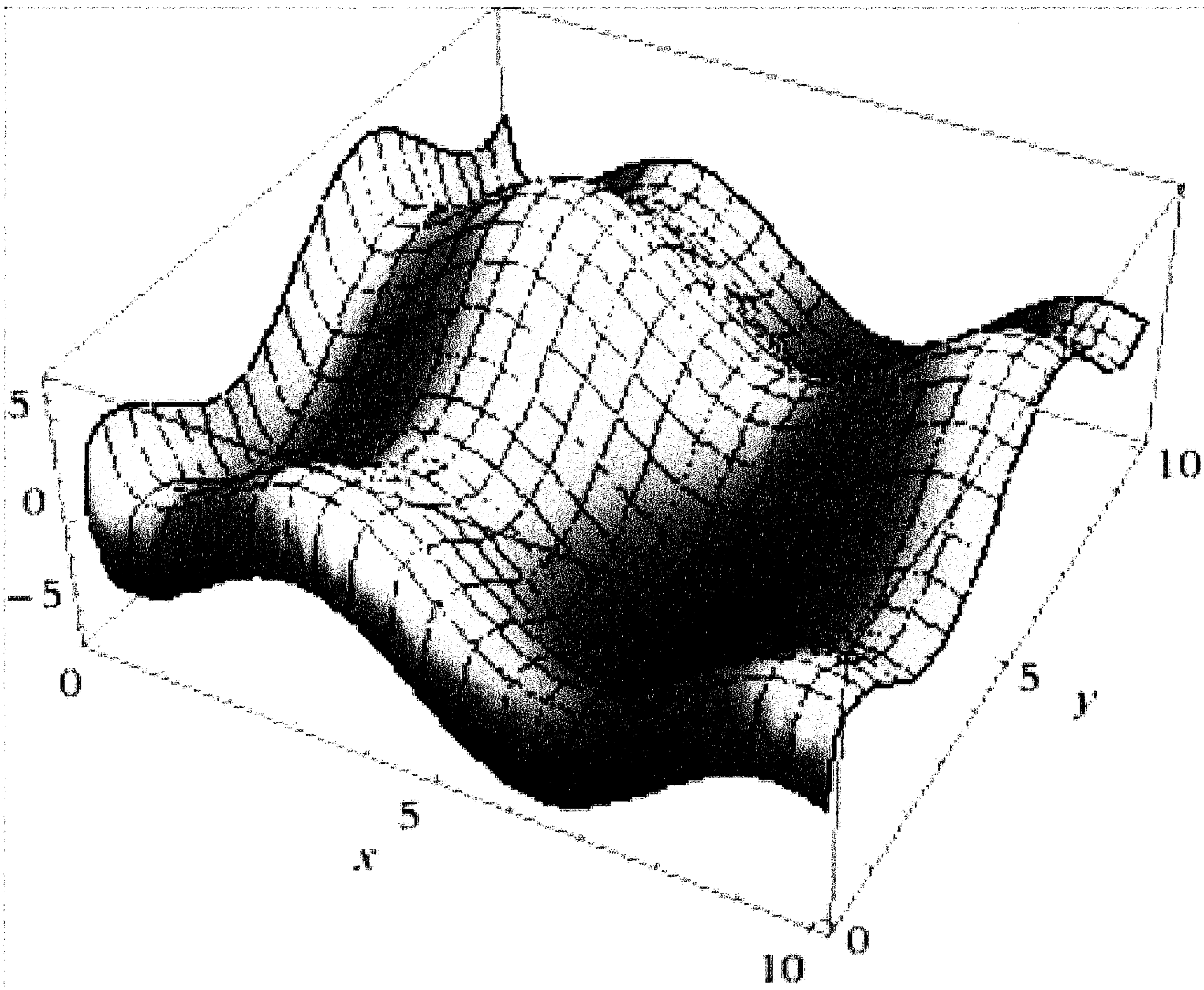
$f\left(\frac{3}{2}, 4\right) = 4^2 - \left(\frac{3}{2}\right)^2 - 3(4) + 3\left(\frac{3}{2}\right) = 16 - \frac{9}{4} - 12 + \frac{9}{2}$

$= \underline{\frac{25}{4}}$

$f\left(0, \frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) = \frac{9}{4} - \frac{9}{2} = \frac{9-18}{4} = \underline{-\frac{9}{4}}$

6. $f(x,y) = \int_x^y \left(\frac{1}{t} - 3\sin t\right) dt = \ln t + 3\cos t \Big]_x^y$

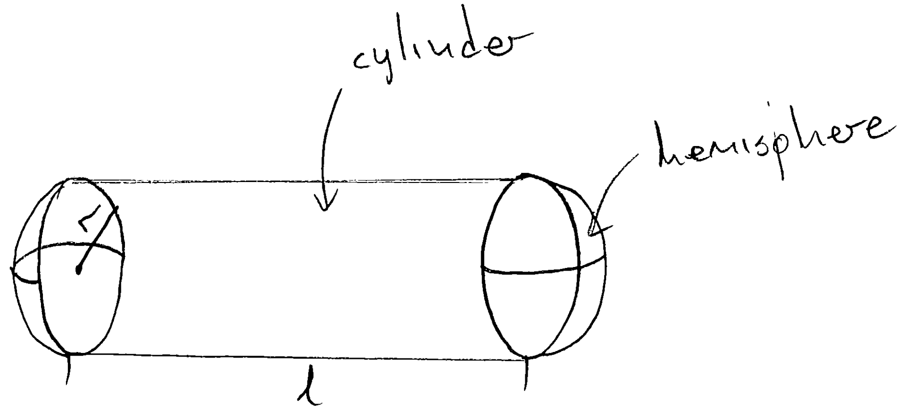
$= \ln y - \ln x + 3\cos y - 3\cos x$



5
0
-5
0
5
x
10
0
5
y
10

(5)

7.



Volume = Volume cylinder + Volume Sphere

$= \pi r^2 l + \frac{4}{3}\pi r^3$

$\underline{V(r,l) = \pi r^2\left(l + \frac{4}{3} r\right)}$

8. $\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y}$ DNE

because approaching (0,0) through the y-axis we obtain $f(0,y)=1$, therefore

$f(x,y) \to 1$ as $(x,y)\to(0,0)$

If we approach (0,0) along the trajectory $y=x$ we obtain $f(x,x) = \frac{2x}{x^2+x} = \frac{2}{x+1}$, therefore

$f(x,y) \to 2$ as $(x,y)\to(0,0)$

This means that the limit DNE.

9. $\lim_{(x,y)\to(1,1)} \frac{xy-1}{xy+1}$

Lines of different slope that pass through $(1,1)$ are given by $y = m(x-1)+1$, so if we take one of those lines to approach $(1,1)$, we can transform

$$\frac{xy-1}{xy+1} = \frac{x(mx-m+1)-1}{x(mx-m+1)+1} = \frac{mx^2-mx+x-1}{mx^2-mx+x+1}$$

as $x \to 1$, we obtain $\frac{m-m+1-1}{m-m++1} = 0$ for all values of m.

We can conclude that

$$\lim_{(x,y)\to(1,1)} \frac{xy-1}{xy+1} = 0$$

10. $$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2}$$

Approaching (0,0,0) along the x-axis, then

y=0 and z=0 $\Rightarrow$ f(x,y,z) = 0

So

$$\lim_{(x,0,0)\to(0,0,0)} 0 = \underline{0}$$

If we approach (0,0,0) on the xy-plane along

x=y

$$f(x,x,0) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

$$\Rightarrow \lim_{(x,x,0)\to(0,0,0)} \frac{1}{2} = \frac{1}{2}$$

$$\therefore \lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} \quad \text{DNE}$$