Homework #7

Math 243 - Section 50

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1.

If the price of new cars increases, then one can reasonably expect that the number of new cars bought in a year decreases. Therefore,

$$\frac{\partial q_1}{\partial x} < 0.$$

If the price of gas increases, then one would think twice before buying a car. Therefore, it is reasonable to assume that

$$\frac{\partial q_1}{\partial y} < 0,$$

because as y increases, q_1 decreases.

However, one could also argue that since new cars consume less fuel than old cars, then as the price of gas increases, the number of bought cars increases. If you are in favor of this argument, then

$$\frac{\partial q_1}{\partial y} > 0.$$

The effects of the price of new cars on the amount of gas sold are harder to determine. For instance, one could argue that since $\frac{\partial q_1}{\partial x} < 0$ there are fewer new, fuel-efficient cars on the road and therefore older cars continue to be used, which means that the quantity of gasoline remains the same, and thus

$$\frac{\partial q_2}{\partial x} = 0$$

that is, there is no effect on the gas quantity in use.

If the price of gas increases, there is clearly a drop in demand, thus

$$\frac{\partial q_2}{\partial y} < 0$$

2. The most commonly cited example is

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

First note that $f(x,y)$ is continuous at $(0,0)$ because

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$ [Verify this trying the approach $y=mx$]

Differentiating $f(x,y)$: [Verify this]

$$f_x(x,y) = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} = u(x,y)$$

$$f_y(x,y) = \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} = v(x,y)$$

Again, these functions are continuous at $(0,0)$.

Now, by definition

$$f_{xy}(x,y) = u_y(x,y) = \lim_{h\to 0} \frac{u(x,y+h) - u(x,y)}{h}$$

and $$f_{xy}(0,0) = u_y(0,0) = \lim_{h\to 0} \frac{u(0,h) - u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{-h - 0}{h}$$

$$= \lim_{h\to 0} -1 = -1$$

$$f_{yx}(x,y) = v_x(x,y) = \lim_{h\to 0} \frac{v(x+h,y) - v(0,0)}{h}$$

so $$f_{yx}(0,0) = v_x(0,0) = \lim_{h\to 0} \frac{v(h,0) - v(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h - 0}{h}$$

$$= \lim_{h\to 0} 1 = 1$$

Since the two limits are different, we conclude that $f_{xy} \neq f_{yx}$ at $(0,0)$.

3. $x^2 + 2yz + z^2 = 1$

Differentiating both sides wrt x:

$$2x + 2y\frac{\partial z}{\partial x} + 2z\frac{\partial z}{\partial x} = 0$$

$$2x + (2y + 2z)\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{2y + 2z} = \boxed{\frac{-x}{y+z}}$$

Differentiating both sides wrt y:

$$0 + 2\left(z + y\frac{\partial z}{\partial y}\right) + 2z\frac{\partial z}{\partial y} = 0$$

$$2z + (2y + 2z)\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-2z}{2y + 2z} = \boxed{\frac{-z}{y+z}}$$

4. $f_x(x,y) = 4x^3y^2 - 3y^4$, $f_y(x,y) = 2x^4y - 12xy^3 + 3y^2$

A candidate function $u(x,y)$ would be

$$u(x,y) = \int f_x(x,y)\,dx = x^4y^2 - 3xy^4 + C(y)$$

where $C(y)$ is a function of y that wrt x behaves as a constant.

If we now differentiate $u(x,y)$ wrt y:

$$2x^4y - 12xy^3 + C'(y) = u_y(x,y)$$

By comparing f_y to u_y we see that

$$C'(y) = 3y^2 \Rightarrow C(y) = \int 3y^2\,dy = y^3 + C$$

where C is a constant.

Therefore the function whose partial derivatives are given is:

$$f(x,y) = x^4y^2 - 3xy^4 + y^3 + C$$

5. The plane is given by

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$$

In this case $f(x,y)=e^{x^2-y^2}$ and $x_0=1$, $y_0=-1$.

$$f_x(x,y)=e^{x^2-y^2}(2x)=2xe^{x^2-y^2} \quad @\ x_0,y_0;\ f_x=2$$

$$f_y(x,y)=e^{x^2-y^2}(-2y)=-2ye^{x^2-y^2} \quad @\ x_0,y_0;\ f_y=2$$

So

$$2(x-1)+2(y+1)-(z-1)=0$$

$$2x-2+2y+2-z+1=0$$

$$2x+2y-z+1=0$$

$$\boxed{2x+2y-z=-1}$$

6. First let's find the values for t and u at which $\vec{r}(t)$ and $\vec{p}(u)$ pass through $(2,1,3)$.

$$\vec{r}(t) = \langle 2+3t,\ 1-t^2,\ 3-4t+t^2 \rangle = \langle 2,1,3 \rangle$$

$$\Rightarrow \left.\begin{array}{l} 2+3t=2 \\ 1-t^2=1 \\ 3-4t+t^2=3 \end{array}\right\} \Rightarrow t=0$$

$$\vec{p}(u) = \langle 1+u^2,\ 2u^3-1,\ 2u+1 \rangle = \langle 2,1,3 \rangle$$

$$\Rightarrow \left.\begin{array}{l} 1+u^2=2 \\ 2u^3-1=1 \\ 2u+1=3 \end{array}\right\} \Rightarrow u=1$$

The tangent plane at $(2,1,3)$ contains vectors tangent to $\vec{r}(t)$ and $\vec{p}(u)$, that is, $\vec{r}\,'(0)$ and $\vec{p}\,'(1)$ are in that plane.

$$\vec{r}\,'(t) = \langle 3,\ -2t,\ -4+2t \rangle \Rightarrow \vec{r}\,'(0) = \langle 3,0,-4 \rangle$$

$$\vec{p}\,'(u) = \langle 2u,\ 6u,\ 2 \rangle \Rightarrow \vec{p}\,'(1) = \langle 2,6,2 \rangle$$

With $\vec{r}\,'(0)$ and $\vec{p}\,'(1)$ we can find a normal (5)
vector to the plane:

$$\vec{n} = \vec{r}\,'(0) \times \vec{p}\,'(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix} =$$

$$= 24\hat{i} - (6+8)\hat{j} + 18\hat{k}$$

$$= 24\hat{i} - 14\hat{j} + 18\hat{k}$$

$$= \langle 24, -14, 18 \rangle$$

Thus the equation of the tangent plane is

$$24(x-2) - 14(y-1) + 18(z-3) = 0$$

$$24x - 48 - 14y + 14 + 18z - 54 = 0$$

$$24x - 14y + 18z = 88$$

$$\underline{12x - 7y + 9z = 44}$$

7. A tangent plane to $x^2+4y^2-z^2=4$ would be given by

$$z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

where $z=f(x,y)$. We want to find the values of x_0, y_0, z_0 such that the tangent plane is is parallel to $2x+2y+z=5$.

We need to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. We do so by differentiating $x^2+4y^2-z^2=4$ implicitly:

$$2x+0-2z\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x}=\frac{x}{z}$$

$$0+8y-2z\frac{\partial z}{\partial y}=0$$

$$\frac{\partial z}{\partial y}=\frac{4y}{z}$$

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If the two tangent planes are parallel, their normal vectors are parallel and thus:

$$\left.\begin{aligned} k\left(\frac{x_0}{z_0}\right) &= 2 \\ k\left(\frac{4y_0}{z_0}\right) &= 2 \\ k &= 1 \end{aligned}\right\} \Rightarrow x_0 = 2z_0 = 4y_0$$

These points must satisfy the equation of the hyperboloid:

$$x_0^2 + 4y_0^2 - z_0^2 = 4$$

So

$$x_0^2 + 4\left(\frac{x_0}{4}\right)^2 - \left(\frac{x_0}{2}\right)^2 = 4$$

$$x_0^2 + \frac{x_0^2}{4} - \frac{x_0^2}{4} = 4 \Rightarrow x_0^2 = 4 \text{ or}$$

$$\underline{x_0 = \pm 2}$$

$\therefore$ The points on the paraboloid where the tangent plane is parallel to $2x + 2y + z = 5$ are

$$\left(2, \tfrac{1}{2}, 1\right) \text{ and } \left(-2, -\tfrac{1}{2}, -1\right)$$

8. If $f(x,y) = x^4y^2 + 3x^2 - 2y$, then

$$L(x,y) = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$z_0 = f(x_0,y_0) = f(1,1) = 1 + 3 - 2 = 2$$

$$f_x(x,y) = 4x^3y^2 + 6x \Rightarrow f_x(1,1) = 4 + 6 = 10$$

$$f_y(x,y) = 2x^4y - 2 \Rightarrow f_y(1,1) = 2 - 2 = 0$$

So $L(x,y) = 2 + 10(x-1) + 0(y-1)$

$$= 2 + 10x - 10 = \underline{10x - 8}$$

$$L(0.98, 1.05) = 9.8 - 8 = \underline{1.8}$$

$$f(0.98, 1.05) = 1.7981$$

$$L(0.98, 1.05) \approx f(0.98, 1.05)$$

9. Let the four numbers be labeled as a, b, c, d respectively. Then their product is

$$P(a,b,c,d) = abcd$$

The error in the computed product can be estimated by

$$dP = \frac{\partial P}{\partial a}\,da + \frac{\partial P}{\partial b}\,db + \frac{\partial P}{\partial c}\,dc + \frac{\partial P}{\partial d}\,dd$$

$$= bcd\,(da) + acd\,(db) + abd\,(dc) + abc\,(dd)$$

Since $da = db = dc = dd = 0.05$ and the maximum value for a, b, c, d is 50 we can write:

$$dP \leq 4 \cdot 50^3 (0.05)$$

$$dP \leq 25000$$

10. The volume of a right circular cylinder is given by

$$V(r,h) = \pi r^2 h$$

So
$$dV = \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial h}\,dh$$
$$= 2\pi r h\,(dr) + \pi r^2 (dh)$$

But $dr = 0.04\,r$ and $dh = 0.02\,h$

So $dV = 0.08\pi r^2 h + 0.02\pi r^2 h = 0.1\,\pi r^2 h$

$= 0.1\,V(r,h)$

Maximum error is 10%