Homework #8

Math 243 – Section 50

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1. $z = \sqrt{1+x^2+y^2}$; $x = \ln t$, $y = \cos t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{2}(1+x^2+y^2)^{-\frac{1}{2}}(2x)\left(\frac{1}{t}\right) + \frac{1}{2}(1+x^2+y^2)^{-\frac{1}{2}}(2y)(-\sin t)$$

$$= \frac{x}{\sqrt{1+x^2+y^2}}\left(\frac{1}{t}\right) + \frac{y}{\sqrt{1+x^2+y^2}}(-\sin t)$$

$$= \frac{1}{\sqrt{1+x^2+y^2}}\left(\frac{x}{t} - y\sin t\right)$$

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2. $w = xe^{y/z}$; $x = t^2$, $y = 1-t$, $z = 1+2t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{\partial w}{\partial x} = e^{y/z} \; ; \; \frac{\partial w}{\partial y} = x e^{y/z}\left(\frac{1}{z}\right) \; ; \; \frac{\partial w}{\partial z} = x e^{y/z}\left(-\frac{y}{z^2}\right)$$

$$\frac{dx}{dt} = 2t \; ; \; \frac{dy}{dt} = -1 \; ; \; \frac{dz}{dt} = 2$$

$$\frac{dw}{dt} = e^{y/z}(2t) - \frac{x}{z}e^{y/z} - \frac{2xy}{z^2}e^{y/z}$$

$$= e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

3. $z = \sin\theta \cos\phi \; ; \; \theta = st^2 \; ; \; \phi = s^2 t$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial s}$$

$$= \cos\theta\cos\phi\,(t^2) + (-\sin\theta\sin\phi)(2st)$$

$$= t^2\cos\theta\cos\phi - 2st\sin\theta\sin\phi$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial t}$$

$$= \cos\theta\cos\phi\,(2st) - \sin\theta\sin\phi\,(s^2)$$

$$= 2st\cos\theta\cos\phi - s^2\sin\theta\sin\phi$$

4. $z = e^{x+2y}$; $x = \frac{s}{t}$, $y = \frac{t}{s}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= e^{x+2y}\left(\frac{1}{t}\right) + e^{x+2y}(2)\left(-\frac{t}{s^2}\right)$$

$$= \underline{e^{x+2y}\left(\frac{1}{t} - \frac{2t}{s^2}\right)}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= e^{x+2y}\left(-\frac{s}{t^2}\right) + 2e^{x+2y}\left(\frac{1}{s}\right)$$

$$= \underline{e^{x+2y}\left(\frac{2}{s} - \frac{s}{t^2}\right)}$$

5.

a) $V(l, w, h) = lwh$

$$\frac{dV}{dt} = \frac{\partial V}{\partial l}\frac{dl}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$\frac{dV}{dt} = wh\,l' + lh\,w' + lw\,h'$$

$$\left.\frac{dV}{dt}\right|_{(1,2,2)} = (2)(2)(2) + (1)(2)(2) + (1)(2)(-3) = \underline{6\ m^3/s}$$

b) $s(l,w,h) = 2(lh + wh + lw)$

$$\frac{ds}{dt} = \frac{\partial s}{\partial l}\frac{dl}{dt} + \frac{\partial s}{\partial w}\frac{dw}{dt} + \frac{\partial s}{\partial h}\frac{dh}{dt}$$

$$= 2(h+w)\,l' + 2(h+l)\,w' + 2(l+w)\,h'$$

$$\left.\frac{ds}{dt}\right|_{(1,2,2)} = 2(2+2)(2) + 2(2+1)(2) + 2(1+2)(-3)$$

$$= 16 + 12 - 18 = 10 \ m^2/s$$

c) $L(l,w,h) = \sqrt{l^2 + w^2 + h^2}$

Squaring both sides:

$$L^2 = l^2 + w^2 + h^2$$

Differentiating:

$$2\frac{dL}{dt} = 2l\,l' + 2w\,w' + 2h\,h'$$

$$\frac{dL}{dt} = l\,l' + w\,w' + h\,h'; \text{ at } (1,2,2):$$

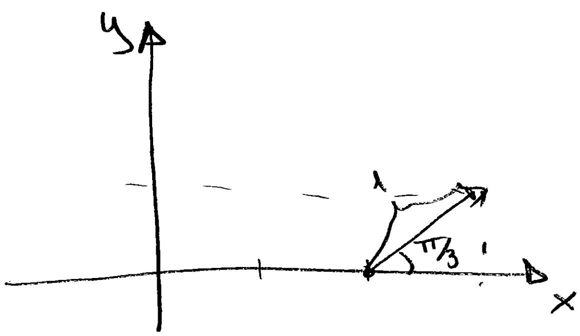
$$= 1(2) + 2(2) + (2)(-3) = \underline{0 \ m/s}$$

6. $f(x,y) = x\sin(xy)$

$$f_x(x,y) = \sin(xy) + x\cos(xy)(y)$$
$$= \sin(xy) + xy\cos(xy)$$

$$f_y(x,y) = x\cos(xy)(x) = x^2\cos(xy)$$

A unit vector with the given characteristics is:



$$\hat{u} = \left\langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \right\rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$\Rightarrow$

$$D_{\hat{u}} f(2,0) = \langle \sin(2(0)) + 2(0)\cos(2(0)), 2^2(\cos(2(0))) \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$= \langle 0, 4 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \underline{2\sqrt{3}}$$

7. $f(x,y) = \frac{y^2}{x}$

$\nabla f(x,y) = \left\langle -\frac{y^2}{x^2}, \frac{2y}{x} \right\rangle$ ✓

$\nabla f(1,2) = \left\langle -\frac{4}{1}, \frac{2(2)}{1} \right\rangle = \langle -4, 4 \rangle$ ✓

$\hat{u} = \left\langle \frac{2}{3}, \frac{\sqrt{5}}{3} \right\rangle$; $\theta = 0$

$D_{\hat{u}} f(1,2) = \langle -4, 4 \rangle \cdot \left\langle \frac{2}{3}, \frac{\sqrt{5}}{3} \right\rangle = \frac{4(\sqrt{5}-2)}{3}$ ✓

8. $\nabla f(x,y,z) = \langle e^y, e^z, e^x \rangle$

@ $(0,0,0)$:

$\nabla f(0,0,0) = \langle 1, 1, 1 \rangle$

$\vec{v} = \langle 5, 1, -2 \rangle \Rightarrow \hat{v} = \frac{1}{\sqrt{30}} \langle 5, 1, -2 \rangle$

$\Rightarrow$

$D_{\hat{v}} f(0,0,0) = \frac{5+1-2}{\sqrt{30}} = \frac{4}{\sqrt{30}}$

9. $\nabla f(x,y) = \langle ye^{-xy}(-y),\ e^{-xy} + ye^{-xy}(-x) \rangle$

$= \langle -y^2 e^{-xy},\ e^{-xy} - xye^{-xy} \rangle$

$\nabla f(0,2) = \langle -4, 1 \rangle$

$\hat{u} = \langle u_x, u_y \rangle \quad ; \quad \|\hat{u}\| = 1 = \sqrt{u_x^2 + u_y^2}$ 1

$\Rightarrow D_{\hat{u}} f(x,y) = \langle -4, 1 \rangle \cdot \langle u_x, u_y \rangle = 1$

$= -4u_x + u_y = 1$ 2

$\Rightarrow u_x^2 + u_y^2 = 1$

$-4u_x + u_y = 1 \Rightarrow u_y = 1 + 4u_x$

$\Rightarrow u_x^2 + (1 + 4u_x)^2 = 1$

$u_x^2 + 1 + 8u_x + 16u_x^2 = 1$

$17u_x^2 + 8u_x + 1 = 1$

$u_x(17u_x + 8) = 0$

$u_x = 0$ or $u_x = \frac{-8}{17}$

and $u_y = 1$ or $u_y = \frac{17}{17} + \frac{4(-8)}{17} = \frac{-15}{17}$

10. If the gradient is perpendicular to the level surface $f(x,y,z) = x^2 - 2y^2 + z^2 + yz = 2$, then one can use $\nabla f(x_0, y_0, z_0)$ as the normal vector of the tangent plane to $f(x,y,z)$ at (x_0, y_0, z_0). So

$$\nabla f(x,y,z) = \langle 2x, -4y+z, 2z+y \rangle$$

$$\nabla f(2,1,-1) = \langle 4, -4(1)-1, 2(-1)+1 \rangle$$

$$= \langle 4, -5, -1 \rangle$$

$\therefore$ An equation of the tangent plane is

$$4(x-2) - 5(y-1) - 1(z+1) = 0$$

or

$$\underline{4x - 5y - z = 4}$$