Homework #9

Math - 243 - Section 50

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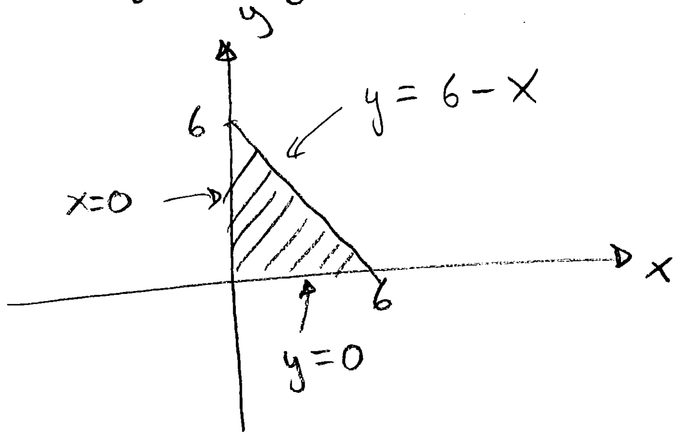
1. $f(x,y) = 4xy^2 - x^2y^2 - xy^3$



$$\nabla f(x,y) = \langle 4y^2 - 2xy^2 - y^3,\ 8xy - 2x^2y - 3xy^2 \rangle$$

If $\nabla f(x,y) = \vec{0}$, then

(1) $4y^2 - 2xy^2 - y^3 = 0$

(2) $8xy - 2x^2y - 3xy^2 = 0$

From (1)

$$4y^2 - y^3 = 2xy^2 \Rightarrow x = \frac{4y^2 - y^3}{2y^2}$$

We can safely assume $y \neq 0$ because we are going to evaluate $f(x,y)$ along the line $y=0$, so

$$x = 2 - \frac{y}{2} \quad (3)$$

(3) in (2):

$$8\left(2-\frac{y}{2}\right)y - 2\left(2-\frac{y}{2}\right)^2 y - 3\left(2-\frac{y}{2}\right)y^2 = 0$$

$$16y - 4y^2 - 2y\left(4 - 2y + \frac{y^2}{4}\right) - 6y^2 + \frac{3}{2}y^3 = 0$$

$$16y - 10y^2 - 8y + 4y^2 - \frac{y^3}{2} + \frac{3}{2}y^3 = 0$$

$$8y - 6y^2 + y^3 = 0$$

$$y(8 - 6y + y^2) = 0 \implies$$

$y = 0$ (discarded because we assumed $y \neq 0$).

or

$y = 2$; $x = 1$

or

$y = 4$; $x = 0$

So, critical points are

$(0,4)$ and $(1,2)$

The Hessian of $f(x,y)$ is

$$Hf(x,y) = \begin{pmatrix} -y^2 & 8y - 4xy - 3y^2 \\ 8y - 4xy - 3y^2 & 8x - 2x^2 - 6xy \end{pmatrix}$$

②

$$Hf(0,4) = \begin{pmatrix} -16 & -16 \\ -16 & 0 \end{pmatrix} \Rightarrow \det(Hf(0,4)) = -(16)^2 < 0$$

$\therefore$ at $(0,4)$ f has a saddle point.

$$Hf(1,2) = \begin{pmatrix} -4 & -4 \\ -4 & -6 \end{pmatrix} \Rightarrow \det(Hf(1,2)) = 24 - 16 > 0$$

$\therefore$ at $(1,2)$ f has a local maximum.

Let's now evaluate the function along the boundaries:

$f(x, y=0) = 0$

$f(x=0, y) = 0$

$y = 6 - x$ or $x = 6 - y$

$$f(6-y, y) = 4(6-y)y^2 - (6-y)^2y^2 - (6-y)y^3$$

$$= 2y^3 - 12y^2$$

$$\Rightarrow f'(6-y, y) = 6y^2 - 24y = 0 \quad \Rightarrow y(6y - 24) = 0$$

$\underline{y = 0}$ or $y = \frac{24}{6} = 4$

So, evaluating the function at all these points (including vertices) and comparing values:

$f(0,4) = 0$ (saddle point)

$f(1, 2) = 4$ (local maximum)

$f(x, 0) = 0$ (boundary)

$f(0, y) = 0$ (boundary)

$f(2, 4) =$ (critical point along $x = 6 - y$)

$= -64$

$f(0,0) = 0$ (vertex)

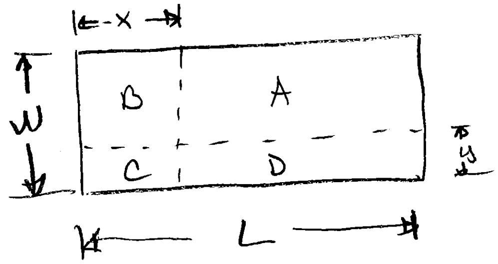
$f(0, 6) = 0$ (vertex)

$f(6,0) = 0$ (vertex)

So, maximum 4 at $(1,2)$

minimum -64 at $(2,4)$

2.



Areas

A: $(L-x)(W-y)$

B: $x(W-y)$

C: xy

D: $(L-x)y$

$$S(x,y) = (L-x)^2(W-y)^2 + x^2(W-y)^2 + x^2y^2 + (L-x)^2y^2$$

$$= x^2\left[(W-y)^2 + y^2\right] + (L-x)^2\left[(W-y)^2 + y^2\right]$$

$$= \left[x^2 + (L-x)^2\right]\left[(W-y)^2 + y^2\right]$$

So, the problem is

$$\min/\max \; S(x,y)$$

subject to

$$0 \le x \le L, \qquad L > 0$$
$$0 \le y \le W, \qquad W > 0$$

$$\nabla S(x,y) = \left\langle \left[(W-y)^2 + y^2\right]\left[2x + 2(L-x)(-1)\right], \right.$$
$$\left. \left[(L-x)^2 + x^2\right]\left[2y + 2(W-y)(-1)\right] \right\rangle$$

$$= \left\langle \left[(W-y)^2 + y^2\right]\left[4x - 2L\right], \right.$$
$$\left. \left[(L-x)^2 + x^2\right]\left[4y - 2W\right] \right\rangle$$

If $\nabla S(x,y) = 0$, then

① $\left[(W-y)^2 + y^2\right]\left[4x - 2L\right] = 0$

② $\left[(L-x)^2 + x^2\right]\left[4y - 2W\right] = 0$

From ①:

$\left[(W-y)^2 + y^2\right] = 0$ or $\left[4x - 2L\right] = 0$

$\downarrow$

Has no real solutions for $W > 0$

$\downarrow$

$4x = 2L$

$$\boxed{x = \frac{L}{2}}$$

Similarly, we can conclude that $y = \frac{W}{2}$ in 2.

Therefore, a critical point is

$$\left(\frac{L}{2}, \frac{W}{2}\right)$$

Evaluating boundaries and critical points:

$$S\left(\frac{L}{2}, \frac{W}{2}\right) = \left[\frac{L^2}{4} + \frac{L^2}{4}\right]\left[\frac{W^2}{4} + \frac{W^2}{4}\right]$$

$$= \left(\frac{L^2}{2}\right)\left(\frac{W^2}{2}\right) = \underline{\frac{L^2W^2}{4}}$$

$$S(x=0, y) = L^2\left((W-y)^2 + y^2\right)$$

$$\Rightarrow S'(x=0, y) = L^2\left[2(W-y)(-1) + 2y\right]$$

$$= L^2\left[-2W + 2y + 2y\right]$$

$$= L^2\left[4y - 2W\right] \Rightarrow \text{If } S'(x=0,y) = 0$$

then $y = \frac{W}{2}$

So $S\left(0, \frac{W}{2}\right) = L^2\left(\frac{W^2}{2}\right) = \underline{\frac{L^2W^2}{2}}$

We can then conclude that

$$S\left(\frac{L}{2}, 0\right) = \frac{L^2 W^2}{2}$$

$$S(0,0) = L^2 W^2 = S(L, W) = S(0, W) = S(L, 0)$$

$\therefore$ The minimum of $S(x,y)$ occurs at

$$\left(\frac{L}{2}, \frac{W}{2}\right)$$

The maximum of $S(x,y)$ occurs when the rectangle is not cut.

3. $d^2 = x^2 + y^2 + z^2$, d = distance from a point (x,y,z) to the origin.

Problem: minimize $d^2 = f(x,y,z)$
subject to
$$xy^2z^3 = 2 \Rightarrow g(x,y,z) = xy^2z^3 - 2 = 0$$

If
$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$\langle 2x, 2y, 2z \rangle = \lambda \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$$

So

$$2x = \lambda y^2 z^3 \qquad (1)$$

$$2y = 2\lambda x y z^3 \qquad (2)$$

$$2z = 3\lambda x y^2 z^2 \qquad (3)$$

$$x y^2 z^3 = 2 \qquad (4)$$

Since $xy^2z^3 = 2$, then $x, y,$ and z cannot be zero, then

$$2x = \lambda y^2 z^3 \qquad (1)$$

$$1 = \lambda x z^3 \qquad (2') \text{ (after dividing by } 2y)$$

$$1 = \frac{3}{2}\lambda x y^2 z \qquad (3') \text{ (after dividing by } 2z)$$

$$x y^2 z^3 = 2 \qquad (4)$$

From (2') an (3'):

$$\cancel{\lambda}\cancel{x} z^3 = \frac{3}{2}\cancel{\lambda}\cancel{x} y^2 z$$

$$2z^3 = 3y^2 z$$

$$2z^2 = 3y^2$$

or

$$y^2 = \frac{2}{3} z^2$$

From (2') and (1)

$$z^3 = \frac{1}{\lambda x}$$

$$2x = \lambda y^2 \left(\frac{1}{\lambda x}\right) = \frac{y^2}{x}$$

$$\Rightarrow 2x^2 = y^2 = \frac{2}{3} z^2$$

$$\Rightarrow x^2 = \frac{z^2}{3}$$

$$\Rightarrow x = \sqrt{\frac{z^2}{3}} = \frac{|z|}{\sqrt{3}} ;$$

Since $xy^2z^3 > 0$, then x and z must have the same sign, so

$$x = \frac{z}{\sqrt{3}}$$

Substituting x and y^2 in (4)

$$\frac{z}{\sqrt{3}}\left(\frac{2}{3}z^2\right)z^3 = 2$$

$$z^6 = 3\sqrt{3} = 3^{3/2} \Rightarrow z = \pm 3^{1/4}$$

$\therefore$ candidate solutions are

$$\left(\pm 3^{-1/2}, (\sqrt{2})3^{-\frac{1}{4}}, \pm 3^{1/4}\right)$$

Both these points have the same function evaluation $\underline{f = 2\sqrt{3}}$

4. The midpoints of the 3x3 array in $[-2,2]\times[-2,2]$ are:

$$\begin{array}{ll}
(0,0) & f(0,0)=0 \\
\left(0,\frac{4}{3}\right) & f\left(0,\frac{4}{3}\right)=0.971 \\
\left(0,-\frac{4}{3}\right) & f\left(0,-\frac{4}{3}\right)=-0.971 \\
\left(\frac{4}{3},0\right) & f\left(\frac{4}{3},0\right)=0 \\
\left(\frac{4}{3},\frac{4}{3}\right) \quad\Rightarrow & f\left(\frac{4}{3},\frac{4}{3}\right)=0.228 \\
\left(\frac{4}{3},-\frac{4}{3}\right) & f\left(\frac{4}{3},-\frac{4}{3}\right)=-0.228 \\
\left(-\frac{4}{3},0\right) & f\left(-\frac{4}{3},0\right)=0 \\
\left(-\frac{4}{3},\frac{4}{3}\right) & f\left(-\frac{4}{3},\frac{4}{3}\right)=0.228 \\
\left(-\frac{4}{3},-\frac{4}{3}\right) & f\left(-\frac{4}{3},-\frac{4}{3}\right)=-0.228
\end{array}$$

$$\Delta A=\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)=\frac{16}{9}$$

Volume ≈ 0

5.

$$\int_{-2}^{2}\int_{-2}^{2} \cos(x)\sin(y)\,dx\,dy$$

$$\int_{-2}^{2} \sin(y)\sin(x)\Big|_{-2}^{2}\,dy$$

$$\int_{-2}^{2} \sin(y)\left(\sin(2)-\sin(-2)\right)dy$$

$$\int_{-2}^{2} \sin(y)\left(2\sin(2)\right)dy = 2\sin(2)\int_{-2}^{2}\sin(y)\,dy$$

$$= 2\sin(2)\left(-\cos(y)\right)\Big|_{-2}^{2} = 2\sin(2)\left(-\cos(2)+\cos(-2)\right)$$

$$= 2\sin(2)\left(-\cos(2)+\cos(2)\right) = \underline{0}$$

6.

$$\int_{0}^{2}\int_{0}^{\pi/2} x\sin(y)\,dy\,dx$$

$$\int_{0}^{2} -x\cos(y)\Big|_{0}^{\pi/2}\,dx$$

$$\int_0^2 \left[-x\cos\left(\frac{\pi}{2}\right) + x\cos(0)\right] dx$$

$$\int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = \underline{2}$$

7. $$\int_0^1 \int_0^3 e^{x+3y}\,dx\,dy$$

$$\int_0^1 e^{x+3y}\Big|_0^3 dy = \int_0^1 \left(e^{3+3y} - e^{0+3y}\right) dy$$

$$= \int_0^1 \left(e^3 e^{3y} - e^{3y}\right) dy = \int_0^1 e^{3y}(e^3-1)\,dy$$

$$= (e^3-1)\int_0^1 e^{3y}\,dy = (e^3-1)\left(\frac{1}{3}e^{3y}\Big|_0^1\right)$$

$$= (e^3-1)\left(\frac{1}{3}(e^3-1)\right)$$

$$= \frac{(e^3-1)^2}{3}$$

8.

$$\int_{-1}^{1}\int_{-y-2}^{y} y^2\,dx\,dy$$

$$\int_{-1}^{1} y^2x\Big|_{-y-2}^{y}\,dy = \int_{-1}^{1} y^2(y+y+2)\,dy$$

$$\int_{-1}^{1}(2y^3+2y^2)\,dy = 2\frac{y^4}{4}+2\frac{y^3}{3}\Big|_{-1}^{1} = \frac{y^4}{2}+\frac{2}{3}y^3\Big|_{-1}^{1}$$

$$= \left(\frac{1}{2}+\frac{2}{3}\right)-\left(\frac{1}{2}-\frac{2}{3}\right) = \frac{7}{6}-\left(\frac{-1}{6}\right) = \frac{8}{6} = \frac{4}{3}$$

9.

$$\int_{1}^{e}\int_{0}^{\ln(x)} x^3\,dy\,dx = \int_{1}^{e} x^3y\Big|_{0}^{\ln x}\,dx = \int_{1}^{e}\ln(x)\,x^3\,dx$$

By parts:

$$u=\ln(x) \qquad v'=x^3$$
$$u'=\frac{1}{x} \qquad v=\frac{x^4}{4}$$

$$\int_{1}^{e}\ln(x)\,x^3\,dx = \frac{\ln(x)\,x^4}{4}\Big|_{1}^{e} - \int_{1}^{e}\frac{x^3}{4}\,dx$$

$$= \left.\frac{\ln(x)\,x^4}{4}\right|_1^e - \left.\frac{x^4}{16}\right|_1^e$$

$$= \left(\frac{\ln(e)\,e^4}{4} - \frac{e^4}{16}\right) - \left(\frac{\ln(1)\,e}{4} - \frac{1}{16}\right)$$

$$= \frac{3e^4}{16} + \frac{1}{16}$$

10. $$\int_0^1\int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 \left.\left(xy + \frac{y^2}{2}\right)\right|_{x^2}^{\sqrt{x}} dx$$

$$\int_0^1 \left[\left(x^{3/2} + \frac{x}{2}\right) - \left(x^3 + \frac{x^4}{2}\right)\right] dx$$

$$\left.\frac{2}{5}x^{5/2} + \frac{x^2}{4} - \frac{x^4}{4} - \frac{x^5}{10}\right|_0^1$$

$$\frac{2}{5} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} - \frac{1}{10} = \frac{3}{10}$$