Name: Marco A. Montes de Oca — MATH 243 - Quiz 1

Section: 50 — September 10, 2012

Please SHOW ALL WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Describe in words the region of $\mathbb{R}^3$ represented by $x^2 + y^2 + z^2 > 2z$.

Solution:

Based on the sum of squares pattern, we can expect a sphere to be involved. To find the center and radius of the sphere we first rearrange the terms:

$$x^2 + y^2 + z^2 - 2z > 0$$

We then complete squares:

$$x^2 + y^2 + (z^2 - 2z + 1) > 1$$

which can be written as:

$$(x - 0)^2 + (y - 0)^2 + (z - 1)^2 > 1$$

The sphere involved in this problem has its center at $(0, 0, 1)$ and has radius equal to one. However, given that the relationship between the two sides of the inequality is determined by $>$, then the region represented by $x^2 + y^2 + z^2 > 2z$ consists of all the points **outside** the aforementioned sphere.

2. (25 pts) Calculate $-2\vec{a} + 3\vec{b}$ and $||\vec{b} - \vec{a}||$ if $\vec{a} = \langle 1, 1, -2\rangle$ and $\vec{b} = \langle 3, -2, 1\rangle$

Solution:

(i) $-2\vec{a} = -2\langle 1, 1, -2\rangle = \langle -2, -2, 4\rangle$, $3\langle 3, -2, 1\rangle = \langle 9, -6, 3\rangle$, so $-2\vec{a} + 3\vec{b} = \langle 7, -8, 7\rangle$.

(ii) $\vec{b} - \vec{a} = \langle 2, -3, 3\rangle$. Therefore, $||\vec{b} - \vec{a}|| = \sqrt{(2)^2 + (-3)^2 + 3^2} = \sqrt{4 + 9 + 9} = \sqrt{22}$.

3. (25 pts) If $\vec{a} = \langle 1, -1 \rangle$, $\vec{b} = \langle 3, 2 \rangle$, and $\vec{c} = \langle 8, -2 \rangle$, find the scalars α and β such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$

   Solution:

   If $\vec{c} = \alpha\vec{a} + \beta\vec{b}$, then $\vec{c} = \alpha\langle 1, -1 \rangle + \beta\langle 3, 2 \rangle = \langle \alpha + 3\beta, -\alpha + 2\beta \rangle = \langle 8, -2 \rangle$. Therefore, $alpha + 3\beta = 8$ (1) and $-\alpha + 2\beta = -2$.

   Solving the system of equations:

   $\alpha = 8 - 3\beta$, thus $-(8 - 3\beta) + 2\beta = -2$, then $5\beta = 6$, or $\beta = \frac{6}{5}$. So, $\alpha = 8 - 3(\frac{6}{5}) = \frac{22}{5}$.

   Result:
   $\beta = \frac{6}{5}$ and $\alpha = \frac{22}{5}$.

4. (25 pts) Find a unit vector $\hat{u}$ parallel to the resultant $\vec{r}$ (that is, the addition vector) of vectors $\vec{r}_1 = \langle 1, 2, 0 \rangle$ and $\vec{r}_2 = \langle 0, 1, -1 \rangle$.

   Solution:

   First, let us find $\vec{r}$. This vector is $\vec{r} = \vec{r}_1 + \vec{r}_2 = \langle 1, 2, 0 \rangle + \langle 0, 1, -1 \rangle = \langle 1, 3, -1 \rangle$.

   The vector $\hat{u} = \frac{\vec{r}}{||\vec{r}||} = \frac{1}{\sqrt{1^2+3^2+(-1)^2}}\langle 1, 3, -1 \rangle = \langle \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}} \rangle$.

Bonus (10 pts): Let C be the point on the line segment between points A and B that is twice as far from A as it is from B. If we denote the origin by O, and let $\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$, and $\vec{c} = \vec{OC}$, show that $\vec{c} = \frac{2}{3}\vec{b} + \frac{1}{3}\vec{a}$.

Solution:

We first write down the given information in vector equation form:

$\vec{a} + \vec{AB} = \vec{b}$, which means that $\vec{AB} = \vec{b} - \vec{a}$ (1). Also, $\vec{a} + \frac{2}{3}\vec{AB} = \vec{c}$ (2). Substituting (1) in (2), we get $\vec{a} + \frac{2}{3}(\vec{b} - \vec{a}) = \frac{1}{3}\vec{a} + \frac{2}{3}\vec{b} = \vec{c}$, which is what we wanted to show.