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Section: 50 | October 8, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Using appropriate differentiation rules, find $\frac{d}{dt}\left[e^{-t}(\vec{v}(t)\cdot\vec{u}(t))\right]$, $\vec{v}(t) = \langle \frac{1}{t}, -1, \ln t\rangle$ and $\vec{u}(t) = \langle t^2, -2t, 1\rangle$.

Solution: $\frac{d}{dt}\left[e^{-t}(\vec{v}(t)\cdot\vec{u}(t))\right] = \left(\frac{d}{dt}e^{-t}\right)(\vec{v}(t)\cdot\vec{u}(t)) + e^{-t}\frac{d}{dt}(\vec{v}(t)\cdot\vec{u}(t)) =$

$\left(\frac{d}{dt}e^{-t}\right)(\vec{v}(t)\cdot\vec{u}(t)) + e^{-t}(\vec{v}\,'(t)\cdot\vec{u}(t) + \vec{v}(t)\cdot\vec{u}\,'(t)) =$

$-e^{-t}(\vec{v}(t)\cdot\vec{u}(t)) + e^{-t}(\vec{v}\,'(t)\cdot\vec{u}(t)) + e^{-t}(\vec{v}(t)\cdot\vec{u}\,'(t))$.

Since $\vec{v}(t)\cdot\vec{u}(t) = \langle \frac{1}{t}, -1, \ln t\rangle \cdot \langle t^2, -2t, 1\rangle = t + 2t + \ln t = 3t + \ln t$, $\vec{v}\,'(t)\cdot\vec{u}(t) = \langle -\frac{1}{t^2}, 0, \frac{1}{t}\rangle \cdot \langle t^2, -2t, 1\rangle = -1 + 0 + \frac{1}{t} = -1 + \frac{1}{t}$, and $\vec{v}(t)\cdot\vec{u}\,'(t) = \langle \frac{1}{t}, -1, \ln t\rangle \cdot \langle 2t, -2, 0\rangle = 2 + 2 = 4$.

$\frac{d}{dt}\left[e^{-t}(\vec{v}(t)\cdot\vec{u}(t))\right] = -e^{-t}(3t+\ln t) + e^{-t}(-1+\frac{1}{t}) + 4e^{-t} = -3te^{-t} - e^{-t}\ln t - e^{-t} + \frac{e^{-t}}{t} + 4e^{-t} = e^{-t}\left(-3t - \ln t + 3 + \frac{1}{t}\right)$.

2. (25 pts) A particle moves along the curve represented by the vector-valued function $\vec{r}(t) = \langle \frac{t^2}{2} - t, 3, \frac{t^2}{2} + 100\rangle$. Find the minimum value of the particle's speed.

Solution: The speed of a particle whose position function is $\vec{r}(t)$, is given by $||\vec{r}\,'(t)||$. Thus the particle's speed in this case is $||\langle t-1, 0, t\rangle|| = \sqrt{(t-1)^2 + t^2}$. The minimum speed is reached when $\frac{d}{dt}\sqrt{(t-1)^2+t^2} = 0$.

$\frac{d}{dt}\sqrt{(t-1)^2+t^2} = \frac{2(t-1)+2t}{2\sqrt{(t-1)^2+t^2}} = \frac{2t-1}{\sqrt{(t-1)^2+t^2}} = 0$. This occurs when $2t - 1 = 0$, that is, when $t = 1/2$. Thus, the minimum speed is $\sqrt{((1/2)-1)^2 + (1/2)^2} = \sqrt{2(1/2)^2} = \sqrt{2}/2$.

3. (25 pts) Find the length of the trajectory followed by a particle whose position function is $\vec{r}(t) = \langle 3, \frac{t^2}{2}, \frac{t^3}{3}\rangle$ from $t = 0$ to $t = 1$.

Solution: The arc length of the particle's trajectory, s, from $t = 0$ to $t = 1$ is given by $s = \int_0^1 ||\vec{r}'(t)||\, dt$.

In this case, $\vec{r}'(t) = \langle 0, t, t^2 \rangle$. So, $||\vec{r}'(t)|| = \sqrt{t^2 + t^4} = t\sqrt{1+t^2}$. Therefore, $s = \int_0^1 t\sqrt{1+t^2}\, dt$.

If $u = 1 + t^2$, $du = 2tdt$, then $s = \int_0^1 t\sqrt{1+t^2}\, dt = \frac{1}{2}\int_1^2 \sqrt{u}\, du = \frac{1}{2}\left.\left(\frac{u^{3/2}}{3/2}\right)\right]_1^2 = \frac{2\sqrt{2}}{3} - \frac{1}{3}$.

4. (25 pts) The trajectories of two particles are given by $\vec{r}(t) = \langle t, t^2, t \rangle$ and $\vec{w}(u) = \langle 3u - 1, 2 + 2u, 1 + u \rangle$, respectively. Do the particles collide?

Solution: If the two particles collide, then there must be a number p, such that $\vec{r}(p) = \vec{w}(p)$. That is, the particles must be at the same point in space *at the same time*.

The collision condition is $\vec{r}(p) = \vec{w}(p)$, thus $\langle p, p^2, p \rangle = \langle 3p - 1, 2 + 2p, 1 + p \rangle$, which means that $p = 3p - 1$ (1), $p^2 = 2 + 2p$ (2), and $p = 1 + p$ (3). From (3), we see that there is no such number because that would imply that $0 = 1$, which is clearly a contradiction. Thus, the two particles cannot possibly collide, even though their trajectories do intersect (verify that).

Bonus (10 pts): Find the curvature of $y = \cos x$ at $x = 0$.

Solution: We need a vector function to represent this curve. This is done by setting $x = t$, $y = \cos x = \cos t$, and $z = 0$ (because the curve is in the xy-plane). We are going to use $\kappa(t) = \frac{||\vec{r}'(t)\times\vec{r}''(t)||}{||\vec{r}'(t)||^3}$ to find the curvature of the graph of $y = \cos x$ at $x = 0$.

$\vec{r}(t) = \langle t, \cos t, 0 \rangle$, $\vec{r}'(t) = \langle 1, -\sin t, 0 \rangle$, $\vec{r}''(t) = \langle 0, -\cos t, 0 \rangle$, and $||\vec{r}'(t)|| = \sqrt{1 + \sin^2 t}$.

$\vec{r}'(t) \times \vec{r}''(t) = (-\cos t)\hat{k}$, which means that $||\vec{r}'(t) \times \vec{r}''(t)|| = |\cos t|$.

Thus, $\kappa(t) = \frac{||\vec{r}'(t)\times\vec{r}''(t)||}{||\vec{r}'(t)||^3} = \frac{|\cos t|}{(\sqrt{1+\sin^2 t})^3}$. At $t = 0$, $\kappa(0) = \frac{1}{(\sqrt{1+0})^3} = 1$.