Name: Marco A. Montes de Oca | MATH 243 - Quiz 5
Section: 50 | November 7, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, if $f(x, y) = ye^x + x$, and $x = u^2 + v$, $y = v - 4u$.

   Solution:

   $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$.
   $\frac{\partial f}{\partial u} = (ye^x + 1)(2u) + e^x(-4) = 2uye^x + 2u - 4e^x = e^x(2uy - 4) + 2u$.

   $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$.
   $\frac{\partial f}{\partial v} = (ye^x + 1)(1) + e^x(1) = ye^x + 1 + e^x = e^x(y + 1) + 1$.

2. (25 pts) Find and classify all the critical points of $f(x, y) = (x - 1 - 2y)(x - 1 + 3y)$.

   Solution:

   $\nabla f(x, y) = \langle (1)(x - 1 + 3y) + (x - 1 - 2y)(1), (-2)(x - 1 + 3y) + (x - 1 - 2y)(3) \rangle = \langle 2x + y - 2, x - 12y - 1 \rangle$.

   At a critical point, $\nabla f(x, y) = \vec{0}$. Therefore, $2x + y - 2 = 0$ and $x - 12y - 1 = 0$ and the only point that satisfies this system of equations is $(1, 0)$. Therefore, the point $(1, 0)$ is the only critical point in the domain of $f(x, y)$.

   To see what kind of critical point $(1, 0)$ is, we need to calculate the determinant of the Hessian matrix of $f(x, y)$. So,

   $H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -12 \end{bmatrix}$ and its determinant is equal to $(2)(-12) - (1)(1) = -25 < 0$. Therefore, at $(1, 0)$ the function has a saddle point.

3. (25 pts) The temperature at the point (x, y) on the surface of a griddle is given by $T(x, y) = 100 - x^2 - y^2$. T is measured in degrees Celsius (°C); x and y, are measured in centimeters (cm). At a certain instant of time, a drop of grease is at the point $(2, 3)$ and is moving toward the point $(2, 2)$ at a rate of $\frac{1}{2}$ centimeters per second (0.5 cm/s). At what rate is the temperature experienced by the drop increasing at that instant? Your answer should be in terms of degrees Celsius per second (°C/s). [Hint: use the Chain Rule.]

Solution:
The drop is at $(2, 3)$ and is moving toward the point $(2, 2)$ at a speed of $\frac{1}{2}$ centimeters per second. A line that describes the path is given by $\vec{p}(t) = \langle 2, 3\rangle + t\langle 2 - 2, 2 - 3\rangle = \langle 2, 3 - t\rangle$. However, the speed of a particle moving according to this function is $||\vec{p}\,'(t)|| = 1$. Therefore, we need to adjust the speed without changing the trajectory of the drop. This can be done multiplying the terms that depend on t in $\vec{p}(t)$ by 1/2. So, the actual position function of the drop of grease is $\vec{r}(t) = \langle 2, 3 - t/2\rangle$.

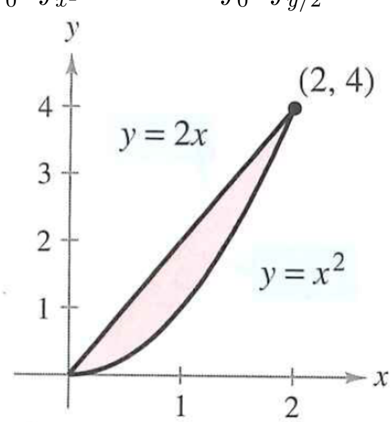
The components of this function describe the x and y coordinates of the drop's position as a function of time. That is, $x(t) = 2$ and $y(t) = 3 - t/2$. With this information we have that

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}.$$

$\frac{dT}{dt} = -2x(0) - 2y(-1/2) = y$. Therefore, at $(2, 3)$ $\frac{dT}{dt} = 3$ °C/s.

4. (25 pts) Give a geometric argument for the given equality. Verify the equality analytically.

$$\int_0^2 \int_{x^2}^{2x} dy\, dx = \int_0^4 \int_{y/2}^{\sqrt{y}} dx\, dy.$$



Solution: These iterated integrals are equal because they are equivalent to $\iint_R dA$, where R is the region shaded in the figure. That is, their evaluation is equal to the area of the shaded area.

$$\int_0^2 \int_{x^2}^{2x} dy\, dx = \int_0^2 y\Big]_{x^2}^{2x} dx = \int_0^2 (2x - x^2)\, dx = x^2 - x^3/3\Big]_0^2 = (2)^2 - (2)^3/3 = 4 - 8/3 =$$
$(12 - 8)/3 = 4/3$.

$$\int_0^4 \int_{y/2}^{\sqrt{y}} dx\, dy = \int_0^4 x\Big]_{y/2}^{\sqrt{y}} dy = \int_0^4 (\sqrt{y} - y/2)\, dy = \frac{y^{3/2}}{3/2} - \frac{y^2}{4}\Bigg]_0^4 = \frac{2}{3}y^{3/2} - \frac{y^2}{4}\Bigg]_0^4 = \frac{2}{3}4^{3/2} - 4 =$$
$16/3 - 4 = (16 - 12)/3 = 4/3$.

Bonus (10 pts): Is it true that if $f(x, y) = 1 - x^2 - y^2$, then $D_{\hat{u}} f(0, 0) = 0$ for any unit vector $\hat{u}$? Explain why or why not.

Solution: Yes, it is true. The reason is that at $(0, 0)$ the function $f(x, y)$ has a local maximum. Therefore, since $\nabla f(0, 0) = \vec{0}$, then $D_{\hat{u}} f(0, 0) = \nabla f(0, 0) \cdot \hat{u} = 0$, for any $\hat{u}$.