University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
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Solution Exam I

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Problems

1. [20 points] Suppose $\vec{a} \cdot \vec{b} = 8$ and $\vec{a} \times \vec{b} = 12\hat{i} - 3\hat{j} + 4\hat{k}$. If the angle between $\vec{a}$ and $\vec{b}$ is θ, find $\tan\theta$.

   Solution:

   Considering that $\vec{a} \cdot \vec{b} = ||\vec{a}||||\vec{b}|| \cos\theta = 8$ and that $||\vec{a} \times \vec{b}|| = ||\vec{a}||||\vec{b}|| \sin\theta$, then $\tan\theta = \frac{||\vec{a}\times\vec{b}||}{\vec{a}\cdot\vec{b}}$.

   Thus, $\tan\theta = \frac{||\vec{a}\times\vec{b}||}{\vec{a}\cdot\vec{b}} = \frac{\sqrt{(12)^2+(-3)^2+(4)^2}}{8} = \frac{\sqrt{144+9+16}}{8} = \frac{13}{8}$.

2. [20 points] For what values of t are $\langle 2t, t, -1\rangle$ and $\langle 4, 2, 1\rangle$ orthogonal?

   Solution:

   If $\langle 2t, t, -1\rangle$ and $\langle 4, 2, 1\rangle$ are orthogonal, then $\langle 2t, t, -1\rangle \cdot \langle 4, 2, 1\rangle = 0$.

   This means that $8t + 2t - 1 = 0 \rightarrow t = \frac{1}{10}$.

3. [20 points] Find an equation of the plane that contains the lines $\vec{r}_1 = \langle 1, 1, 0\rangle + t\langle 1, -1, 2\rangle$ and $\vec{r}_2 = \langle 2, 0, 2\rangle + s\langle -1, 1, 0\rangle$.

   Solution:

Since the two lines lie in the same plane, the cross product of their direction vectors is perpendicular to the plane, a necessary ingredient to define a plane. Thus, if $\vec{a} = \langle 1, -1, 2 \rangle$ and $\vec{b} = \langle -1, 1, 0 \rangle$. then

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = -2\hat{i} - 2\hat{j} + 0\hat{k} = \langle -2, -2, 0 \rangle.$$

And a vector equation of the plane is given by $(\vec{r} - \vec{u}) \cdot \vec{n} = 0$ where $\vec{u} = \langle 1, 1, 0 \rangle$. Then, a scalar equation of the plane is $-2(x-1) - 2(y-1) + 0(z-0) = 0 \to x + y = 2$.

4. [20 points] Find a unit vector that is perpendicular to both $\vec{a} = \langle 3, 2, 0 \rangle$ and $\vec{b} = \langle -1, 0, 3 \rangle$.

Solution:

Such a vector is parallel to $\vec{n} = \vec{a} \times \vec{b}$. Now, given that

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\hat{i} - 9\hat{j} + 2\hat{k} = \langle 6, -9, 2 \rangle.$$

The vector we are looking for is $\hat{n} = \frac{\vec{n}}{||\vec{n}||} = \frac{1}{\sqrt{36+81+4}} \langle 6, -9, 2 \rangle = \frac{1}{11} \langle 6, -9, 2 \rangle$.

5. [20 points] Find the area of the parallelogram spanned by $\vec{a} = \langle -1, 3, 1 \rangle$ and $\vec{b} = \langle 2, 0, 1 \rangle$.

Solution:

The area spanned by $\vec{a} = \langle -1, 3, 1 \rangle$ and $\vec{b} = \langle 2, 0, 1 \rangle$ is equal to $||\vec{a} \times \vec{b}||$. Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 3\hat{i} + 3\hat{j} - 6\hat{k} = \langle 3, 3, -6 \rangle.$$

Thus, the area is equal to $||\vec{a} \times \vec{b}|| = \sqrt{9 + 9 + 36} = \sqrt{54}$.

[Bonus problem: 10 points] If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors, does $\vec{a} \cdot \vec{b} = \vec{c} \cdot \vec{b}$ imply that $\vec{a} = \vec{c}$? Show that it is true in general, or disprove by providing an appropriate example.

Solution:

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors, $\vec{a} \cdot \vec{b} = \vec{c} \cdot \vec{b}$ does not imply that $\vec{a} = \vec{c}$.

This is shown through an example: Let $\vec{a} = \langle 1, 0, 0 \rangle$, $\vec{b} = \langle 1, 1, 1 \rangle$, and $\vec{c} = \langle 0, 1, 0 \rangle$. Then $\vec{a} \cdot \vec{b} = \vec{c} \cdot \vec{b} = 1$; however, $\vec{a} \neq \vec{c}$.