University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
Instructor: Dr. Marco A. MONTES DE OCA
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Solution Exam II

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Problems

1. [20 points] An ant crawls along the radius from the center to the edge of a circular disk of radius 1 meter, moving at a constant rate of 1 cm/s. Meanwhile, the disk is turning counterclockwise about its center at 1 revolution per second. What is the vector function that describes the motion of the ant?

   Solution: As the ant moves, the disk rotates. Thus, the trajectory of the ant is a spiral that extends from the center of the disk to its edge.

   If the ant were static at, say (2,0), its trajectory would be a circle represented by $\vec{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$. But we are told that the ant moves and that every cm. walked by the ant will coincide with one complete revolution. So, assuming that the center of the disk is at (0,0), after 1 second, the ant will be at the point (1,0), after 2 seconds, the ant will be at (2,0) and so on. This means that radius of the spiral grows linearly with time. Thus, the basic form of the vector function that describes the ant's trajectory is $\vec{r}(t) = \langle t\cos(t), t\sin(t)\rangle$.

   We now need to adjust the rate at which the ant turns. In its current form, the vector function does describe a spiral but the ant would have to wait 2π seconds to give a complete revolution. We are told that the revolution occurs every second; thus, if we multiply t by 2π we will satisfy that requirement.

   Therefore, the vector function that describes the ant's trajectory is $\vec{r}(t) = \langle t\cos(2\pi t), t\sin(2\pi t)\rangle$, for $0 \leq t \leq 100$.

2. [20 points] Find the length of the curve $\vec{r}(t) = \langle 2t, t^2, \frac{t^3}{3}\rangle$, $0 \leq t \leq 1$.

Solution: We are going to use the formula $s = \int_a^b ||\vec{r}'(t)||\, dt$.

$\vec{r}'(t) = \langle 2, 2t, t^2 \rangle$, thus $||\vec{r}'(t)|| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2+t^2)^2} = |2 + t^2|$. Since $t^2 > 0$, $|2 + t^2| = 2 + t^2$.

So, $s = \int_0^1 (2 + t^2)\, dt = 2t + \frac{t^3}{3}\Big]_0^1 = 2 + \frac{1}{3} = \frac{7}{3}$.

3. [20 points] Find the unit tangent and the principal unit normal vectors for the curve $\vec{r}(t) = \langle 4\cos t, 4\sin t, 3t \rangle$ at $t = \pi/2$.

   Solution: The unit tangent vector is defined as $\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||}$ and the principal unit normal vector as $\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||}$.

   $\vec{r}'(t) = \langle -4\sin t, 4\cos t, 3 \rangle$ and $||\vec{r}'(t)|| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = \sqrt{16 + 9} = \sqrt{25} = 5$. Therefore,

   $\vec{T}(t) = \frac{1}{5}\langle -4\sin t, 4\cos t, 3 \rangle$. Then, $\vec{T}(\pi/2) = \langle -\frac{4}{5}, 0, \frac{3}{5} \rangle$.

   Now, $\vec{T}'(t) = \frac{1}{5}\langle -4\cos t, -4\sin t, 0 \rangle$ and $||\vec{T}'(t)|| = \frac{1}{5}\sqrt{16\cos^2 t + 16\sin^2 t} = \frac{1}{5}\sqrt{16} = \frac{4}{5}$. Therefore,

   $\vec{N}(t) = \frac{1}{5}\frac{5}{4}\langle -4\cos t, -4\sin t, 0 \rangle = \frac{1}{4}\langle -4\cos t, -4\sin t, 0 \rangle = \langle -\cos t, -\sin t, 0 \rangle$. Then, $\vec{N}(\pi/2) = \langle 0, -1, 0 \rangle$.

4. [20 points] Find the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy\cos y}{3x^2 + y^2}$.

   Solution: Approaching (0,0) from the x- or y-axis, results in the limit being 0. If we approach the origin via the line $x = y$, we obtain:

   $$\lim_{(x,x)\to(0,0)} \frac{x^2\cos x}{3x^2 + x^2} = \lim_{(x,x)\to(0,0)} \frac{\cos x}{4} = \frac{1}{4}.$$

   Since this limit and the ones obtained previously are different, we can conclude that the limit does not exist.

5. [20 points] Find all the second order partial derivatives of $f(x, y) = \dfrac{x}{xy + 1}$.

   Solution:

   $f_x(x, y) = \frac{(xy+1) - xy}{(xy+1)^2} = \frac{1}{(xy+1)^2}$.
   $f_y(x, y) = \frac{(xy+1)(0) - x(x)}{(xy+1)^2} = \frac{-x^2}{(xy+1)^2}$.

   $f_{xy}(x, y) = -2(xy + 1)^{-3}(x) = \frac{-2x}{(xy+1)^3}$.
   $f_{yx}(x, y) = \frac{(xy+1)^2(-2x) - (-x^2)(2(xy+1)y)}{(xy+1)^4} = \frac{-2x}{(xy+1)^3}$.
   $f_{xx}(x, y) = -2(xy + 1)^{-3}(y) = \frac{-2y}{(xy+1)^3}$.
   $f_{yy}(x, y) = -x^2(-2)(xy + 1)^{-3}(x) = \frac{2x^3}{(xy+1)^3}$.

[Bonus problem: 10 points] Find an equation of the tangent plane to $z = \sqrt{xy}$ at $(1, 1, 1)$.

Solution:

$f_x(x, y) = \frac{1}{2}(xy)^{-1/2}(y) = \frac{y}{2\sqrt{xy}}$. Then $f_x(1, 1) = \frac{1}{2}$

$f_y(x, y) = \frac{1}{2}(xy)^{-1/2}(x) = \frac{x}{2\sqrt{xy}}$. Then $f_y(1, 1) = \frac{1}{2}$

The equation of the plane at $(1, 1, 1)$ is therefore:

$\frac{1}{2}(x - 1) + \frac{1}{2}(y - 1) - (z - 1) = 0$ or $\frac{x}{2} + \frac{y}{2} - z = 0$ or $x + y - 2z = 0$.