University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
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Solution Exam III

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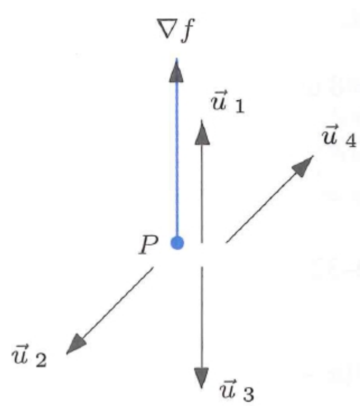
Problems

1. [20 points] The vector $\nabla f(x, y)$ at a point P and four unit vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, and $\vec{u}_4$ are shown in the figure below. Arrange the following quantities in ascending order. Explain your reasoning.

$$D_{\vec{u}_1} f(x, y),\ D_{\vec{u}_2} f(x, y),\ D_{\vec{u}_3} f(x, y),\ D_{\vec{u}_4} f(x, y),\ 0.$$

The directional derivatives are all evaluated at the point P and the function $f(x, y)$ is differentiable at P.



Solution: Since the directional derivative of a function $f(x, y)$ at a point (x_0, y_0) in the direction of a unit vector $\hat{u}$ is given by $D_{\vec{u}_1} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}$, we can conclude from the figure that

$$D_{\vec{u}_3} f(x, y) < D_{\vec{u}_2} f(x, y) < 0 < D_{\vec{u}_4} f(x, y) < D_{\vec{u}_1} f(x, y).$$

2. [20 points] The temperature at a point (x, y) on a metal plate is modeled by

$$T(x, y) = e^{-(x^2+2y^2)}$$

Find directions of no change in heat on the plate from the point $(1, 1)$. [The answer should be at least one unit vector that gives the direction of movement from $(1, 1)$ such that no change in temperature is felt.]

Solution: Since the directional derivative in a direction perpendicular to the gradient is zero, then we just need to find a unit vector perpendicular to the gradient $\nabla f(1, 1)$.

$\nabla f(x, y) = \langle e^{-(x^2+2y^2)}(-2x), e^{-(x^2+2y^2)}(-4y)\rangle = -2e^{-(x^2+2y^2)}\langle x, 2y\rangle$
$\nabla f(1, 1) = -2e^{-(1+2)}\langle 1, 2\rangle = -2e^{-3}\langle 1, 2\rangle$

Therefore, one vector perpendicular to $\nabla f(1, 1)$ is $\vec{u} = \langle -2, 1\rangle$. Normalizing, we obtain the vector $\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$. The other unit vector that meets the requirements is $\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\rangle$

3. [20 points] Use Lagrange multipliers to prove that the product of three positive numbers x, y, and z, whose sum has the constant value S, is a maximum when the three numbers are equal. Use this result to prove that

$$\sqrt[3]{xyz} \leq \frac{x+y+z}{3}.$$

Solution: The first part is to maximize $f(x, y, z) = xyz$ subject to $x + y + z = S$. Using Lagrange multipliers, we obtain:

$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$
$\langle yz, xz, xy\rangle = \lambda\langle 1, 1, 1\rangle$

The system of equation is therefore

$yz = \lambda$ (1)
$xz = \lambda$ (2)
$xy = \lambda$ (3)
$x + y + z = S$ (4)

Since $x > 0$, $y > 0$ and $z > 0$, from (1) and (2) we conclude that $x = y$, and from (2) and (3) we conclude that $y = z$. Thus, $x = y = z$ (5), which is what we wanted to show. Note that this solution maximizes $f(x, y, z)$ because the minimum would be a solution with say $x \to S$, $y \to 0$ and $z \to 0$.

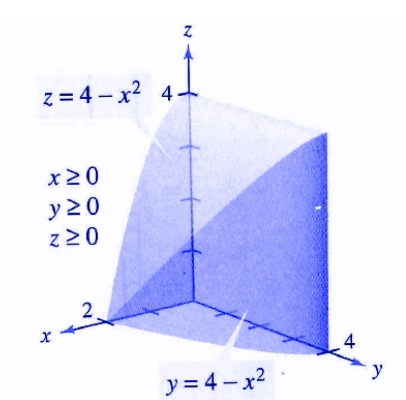
For the second part, if we consider that when $x = y = z = \frac{S}{3}$ the product xyz is maximum, then in general:

$xyz \leq (\frac{S}{3})^3$

taking the cubic root to both sides of the equation: $\sqrt[3]{xyz} \leq \frac{S}{3}$

but since $x + y + z = S$, we conclude that $\sqrt[3]{xyz} \leq \frac{x+y+z}{3}$.

4. [20 points] Use a triple integral to find the volume of the solid shown in the figure.



Solution: The volume of this solid can be calculated as:

$V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2} \, dz \, dy \, dx$
$V = \int_0^2 \int_0^{4-x^2} (4-x^2) \, dy \, dx$
$V = \int_0^2 4y - x^2y\Big|_0^{4-x^2} \, dx = \int_0^2 (4(4-x^2) - x^2(4-x^2)) \, dx = \int_0^2 (16 - 4x^2 - 4x^2 + x^4) \, dx = \int_0^2 (16 - 8x^2 + x^4) \, dx$
$V = 16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\Big|_0^2 = 32 - \frac{64}{3} + \frac{32}{5} = \frac{480-320+96}{15} = \frac{256}{15}$

5. [20 points] A bead is made by drilling a cylindrical hole of radius 1mm through a sphere of radius 5mm. Find the volume of the bead using a triple integral in cylindrical coordinates.

Solution: Place the bead's center at the origin, and let the cylindrical hole be parallel to the z-axis. Let the volume of the bead be V. Then by symmetry $\frac{V}{2}$ is the volume of the upper part of the bead. The volume of this upper part can be calculated by

$\frac{V}{2} = \int_0^{2\pi} \int_1^5 \int_0^{\sqrt{25-r^2}} r \, dz \, dr \, d\theta$
$\frac{V}{2} = \int_0^{2\pi} \int_1^5 \sqrt{25-r^2} r \, dr \, d\theta$

Using $u = 25 - r^2$, then $du = -2r\, dr$, so
$\frac{V}{2} = -\frac{1}{2} \int_0^{2\pi} \int_{24}^0 \sqrt{u} \, du \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^{24} \sqrt{u} \, du \, d\theta$
$\frac{V}{2} = \frac{1}{2} \int_0^{2\pi} \frac{2}{3} u^{3/2}\Big|_0^{24} \, d\theta = \frac{1}{3} \int_0^{2\pi} (24)^{3/2} \, d\theta$
$\frac{V}{2} = \frac{(24)^{3/2}}{3}(2\pi)$, which means that $V = \frac{4(24)^{3/2}\pi}{3} = 64\sqrt{6}\pi$.

[Bonus problem: 10 points] Find and classify all the critical points of $f(x, y) = -x^4 - y^4 + 4xy - 2$.

Solution:
$f_x = -4x^3 + 4y$
$f_y = -4y^3 + 4x$

If $f_x = f_y = 0$, then
$x^3 = y$ (1)
$y^3 = x$ (2)

(1) in (2):
$(x^3)3 = x$
$x^9 - x = 0$
$x(x^8 - 1) = 0$

So $x = 0$ or $x = \pm 1$. Therefore, there are three critical points: $(0, 0)$, $(1, 1)$. and $(-1, -1)$.

$f_{xx} = -12x^2$
$f_{yy} = -12y^2$
$f_{xy} = f_{yx} = 4$

So, $\det(H)(x, y) = 144x^2y^2 - 16$.

For $(0, 0)$, $\det(H)(0, 0) < 0$, so $(0, 0)$ is a saddle point.
For $(1, 1)$, $\det(H)(1, 1) > 0$, and $f_{xx} < 0$ so at $(1, 1)$ f has a local maximum.

For $(-1,-1)$, $\det(H)(-1,-1) > 0$, and $f_{xx} < 0$ so at $(-1,-1)$ f has a local maximum.