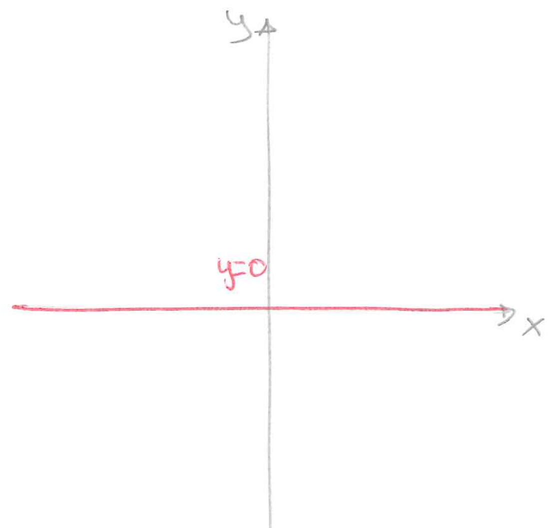
Homework #1

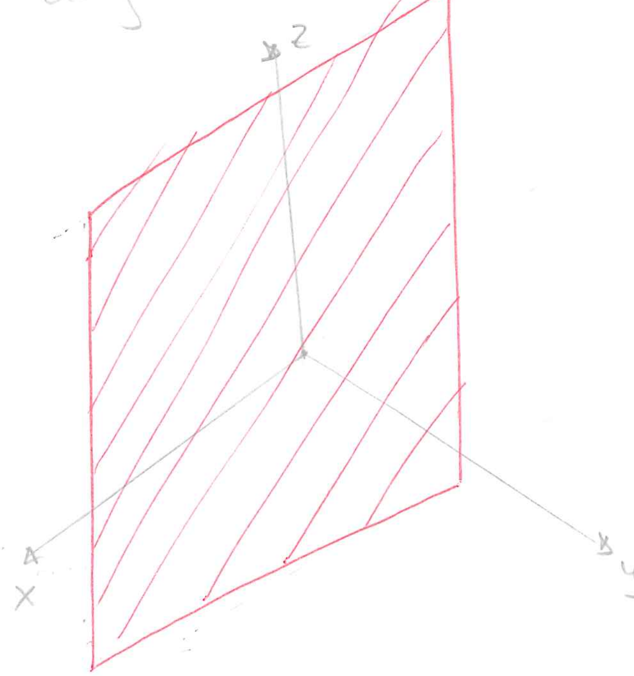
Math 243 - Section 51

Dr. Marco A. Montes de Oca

1. a) $y=0$ in $\mathbb{R}^2$ represents the x-axis, as can be seen in the following figure



b) $y=0$ in $\mathbb{R}^3$ represents the xz-plane. This is because for any value of x and z, $y=0$.



2. a) The distance from $(3,7,-5)$ to any point on the xy-plane $(x,y,0)$ is

$$\sqrt{(x-3)^2+(y-7)^2+(0-(-5))^2} = d$$

However, when we just want the distance to the plane, we want the shortest distance. In this and the following cases, this occurs when the point in question is projected perpendicularly onto the plane. So, the distance from $(3,7,-5)$ to the xy-plane is equal to $|5| = \underline{5}$

b) Using the same reasoning as above, the distance from $(3,7,-5)$ to the yz-plane is equal to $|3| = \underline{3}$

c) As above, the distance from $(3,7,-5)$ to the xz-plane is equal to $|7| = \underline{7}$.

d) Just as we did for the previous cases, let's first find the distance from a point on the x-axis to (3, 7, -5) and then project this point perpendicularly.

The distance from a point on the x-axis to (3, 7, -5) to (x, 0, 0) is

$$\sqrt{(x-3)^2 + (0-7)^2 + (0-(-5))^2} = d$$

Projecting the point perpendicularly to the x-axis we find that the distance from (3, 7, -5) to the x-axis is

$$\sqrt{(-7)^2 + (5)^2} = \sqrt{49+25} = \underline{\sqrt{74} \approx 8.6}$$

e) Using the same reasoning, the distance from (3, 7, -5) to the y-axis is

$$\sqrt{(-3)^2 + (5)^2} = \sqrt{9+25} = \underline{\sqrt{34} \approx 5.83}$$

f) The distance from (3, 7, -5) to the z-axis is

$$\sqrt{(-3)^2 + (-7)^2} = \sqrt{9+49} = \underline{\sqrt{58} \approx 7.61}$$

3. An equation of a sphere with center $(2,-6,4)$ and radius 5 is

$$(x-2)^2+(y-(-6))^2+(z-4)^2=5^2$$

$$(x-2)^2+(y+6)^2+(z-4)^2=25$$

- The intersection with the xy-plane occurs when $z=0$, so

$$(x-2)^2+(y+6)^2+(0-4)^2=25$$

$$(x-2)^2+(y+6)^2+16=25$$

$$(x-2)^2+(y+6)^2=9$$

describes the shape of the intersection. It is a circle centered at $(2,-6)$ with radius 3.

- The intersection with the yz-plane is

$$(y+6)^2+(z-4)^2=25-4=21$$

or a circle centered at $(-6,4)$ on the yz-plane with radius $\sqrt{21}$.

- The intersection with the xz-plane is

$$(x-2)^2+(z-4)^2=25-36=-11$$

Therefore, the sphere does not intersect the xz-plane.

4. For any value of z, the relation $x^2+y^2 \le 4$ must hold. The second step is to restrict the values of z. With the given information, the allowed range of values for z is $[0,8]$, so the inequalities are:

$$\begin{cases} x^2+y^2 \le 4 \\ 0 \le z \le 8 \end{cases}$$

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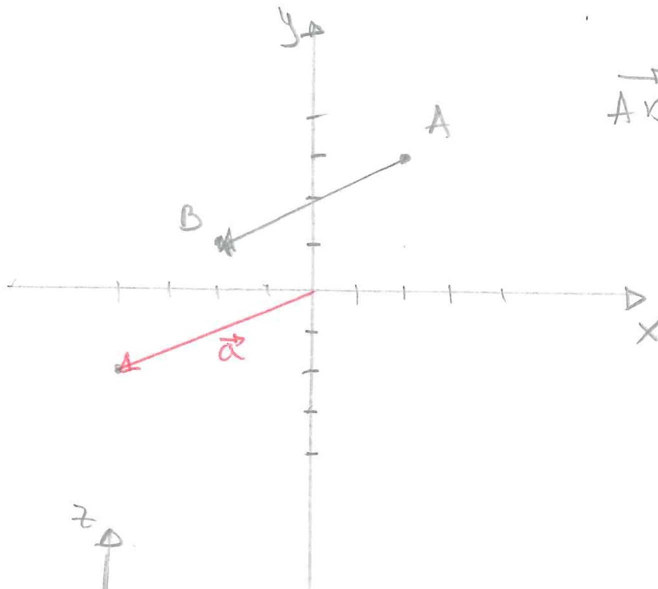
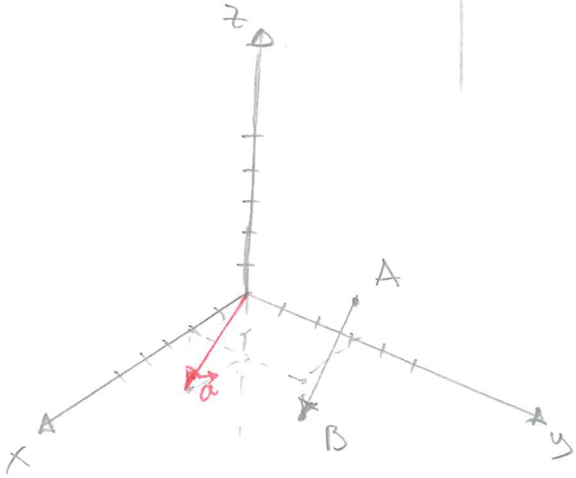
5. a)



$\vec{AB} = \langle -2-2, 1-3 \rangle$
$= \langle -4, -2 \rangle$

b)



$\vec{AB} = \langle 2-0, 3-3, -1-1 \rangle$
$= \langle 2, 0, -2 \rangle$

6.

a) $\vec{a}+\vec{b} = \langle 2,-4,4\rangle + \langle 0,2,-1\rangle = \langle 2+0, 2-4, 4-1\rangle =$

$\underline{\langle 2,-2,3\rangle}$

b) $2\vec{a}+3\vec{b} = 2\langle 2,-4,4\rangle + 3\langle 0,2,-1\rangle = \langle 4,-8,8\rangle + \langle 0,6,-3\rangle$

$\underline{= \langle 4,-2,5\rangle}$

c) $\|\vec{a}\| = \sqrt{(2)^2+(-4)^2+(4)^2} = \sqrt{4+16+16} = \sqrt{36} \underline{= 6}$

d) $\vec{a}-\vec{b} = \langle 2,-4,4\rangle - \langle 0,2,-1\rangle = \langle 2,-6,5\rangle$

$\|\vec{a}-\vec{b}\| = \sqrt{2^2+(-6)^2+5^2} = \sqrt{4+36+25} \underline{= \sqrt{65} \approx 8.06}$

7. Since $\|\alpha\vec{a}\| = |\alpha|\,\|\vec{a}\|$, it would be enough to find a unit vector with the same direction as $\langle -2,5,1\rangle$ and then multiply it by 100.

So, let $\vec{a} = \langle -2,5,1\rangle$, then

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{\langle -2,5,1\rangle}{\sqrt{(-2)^2+5^2+1^2}} = \frac{1}{\sqrt{4+25+1}}\langle -2,5,1\rangle$$

$$= \frac{1}{\sqrt{30}}\langle -2,5,1\rangle$$

Multiplying by 100:

$$\underline{\frac{100}{\sqrt{30}}\langle -2,5,1\rangle}$$

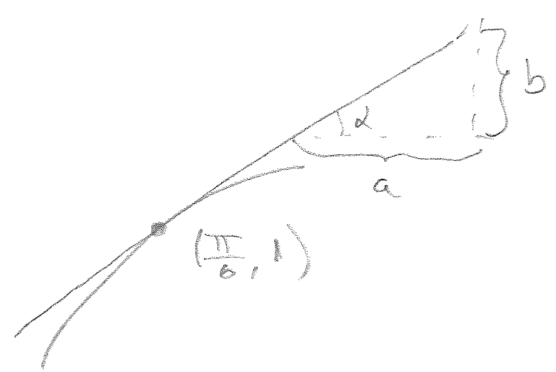
8. To find perpendicular vectors to the tangent line to the curve $y = 2\sin(x)$, we can first find parallel vectors and then rotate them.

This can be done as follows: first let's find the slope of the tangent line at $\left(\frac{\pi}{6}, 1\right)$. This slope is equal to the derivative of $y = 2\sin(x)$ evaluated at $x = \frac{\pi}{6}$.

So

$$y' = 2\cos(x) \; ; \quad y'\Big|_{x=\frac{\pi}{6}} = m = 2\,\frac{\sqrt{3}}{2} = \sqrt{3}$$

Now, the geometric interpretation of slope is the tangent of the inclination angle α:



So $\quad m = \tan\alpha = \dfrac{b}{a} \quad (1)$

Now, we want to find unit vectors, so the components of these vectors must satisfy:

$$\sqrt{a^2 + b^2} = 1 \quad (2)$$

Solving ① and ② simultaneously we obtain:

$$m = \sqrt{3} = \frac{b}{a} \Rightarrow \sqrt{3}\,a = b \quad ①$$

Substituting ① in ②:

$$\sqrt{a^2 + (\sqrt{3}a)^2} = 1$$

$$\sqrt{a^2 + 3a^2} = 1$$

$$\sqrt{4a^2} = 1 \Rightarrow 2|a| = 1 \Rightarrow |a| = \frac{1}{2}$$

or

$$a = \pm\frac{1}{2}$$

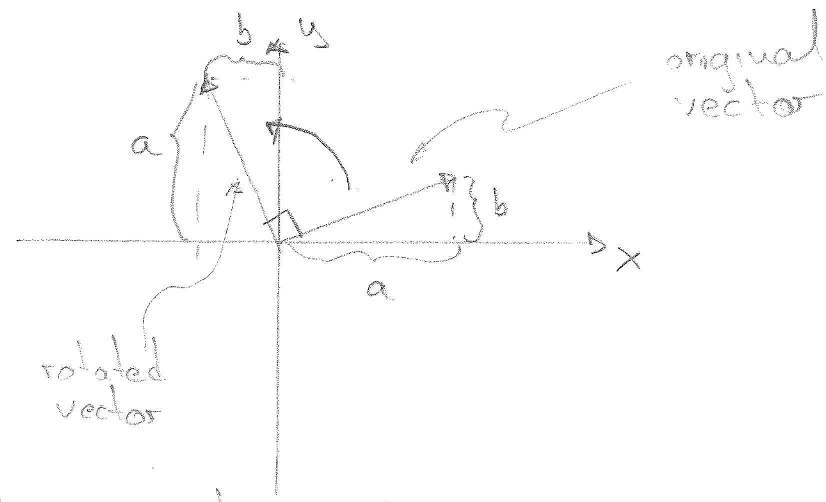
And therefore:

$$b = \pm\frac{\sqrt{3}}{2}$$

So the unit vectors

$\pm\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ are parallel to the tangent line to $y = 2\sin(x)$ at $\left(\frac{\pi}{6}, 1\right)$.

We now need to rotate them. This is accomplished as shown:



So, a perpendicular vector to $\langle a, b\rangle$ is $\langle -b, a\rangle$.

In our case then, the perpendicular vectors we are looking for are:

$$\left\langle \mp \frac{\sqrt{3}}{2}, \pm \frac{1}{2} \right\rangle$$

or

$$\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \text{ and } \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$$

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9. $\vec{c} = s\langle 3,2\rangle + t\langle 2,-1\rangle = \langle 7,1\rangle$ therefore

$\langle 3s+2t, 2s-t\rangle = \langle 7,1\rangle$

$$\begin{cases} 3s+2t=7 & (1) \\ 2s-t=1 & (2) \Rightarrow t = 2s-1 \end{cases}$$

Substituting t in ①:

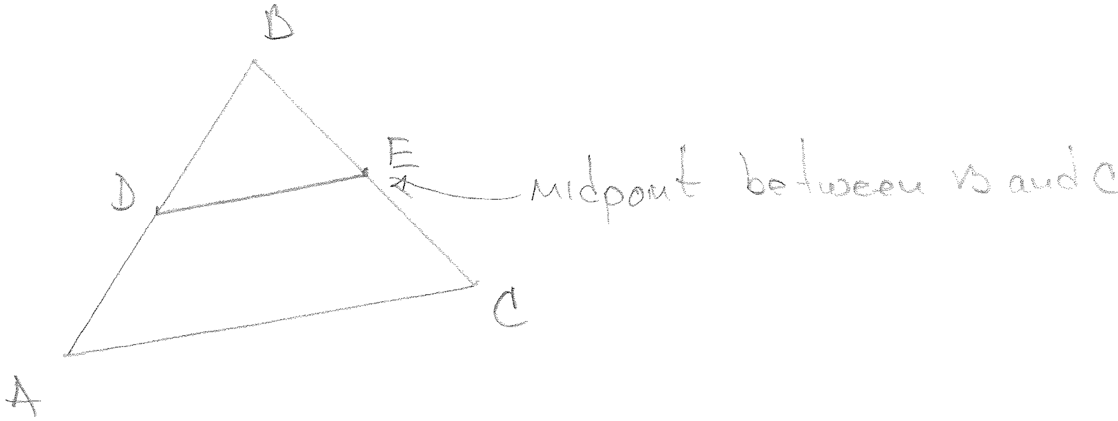
$$3s + 2(2s-1) = 7$$

$$3s + 4s - 2 = 7$$

$$7s - 2 = 7 \Rightarrow 7s = 9 \Rightarrow \boxed{s = \frac{9}{7}}$$

and $t = 2\left(\frac{9}{7}\right) - 1 = \frac{18-7}{7} = \boxed{\frac{11}{7}}$

10. Consider the following figure:



Then:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

also:

$$\overrightarrow{AD} + \overrightarrow{DE} + \overrightarrow{EC} = \overrightarrow{AC}$$

But

$$\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB}$$

$$\overrightarrow{EC} = \frac{1}{2}\overrightarrow{BC}$$

So: $\frac{1}{2}\overrightarrow{AB} + \overrightarrow{DE} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{AC}$

$$\frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) + \overrightarrow{DE} = \overrightarrow{AC}$$

$$\frac{1}{2}\overrightarrow{AC} + \overrightarrow{DE} = \overrightarrow{AC} \Rightarrow \boxed{\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}}$$