Homework #10

Math 243 - Section 51

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1. $f(x,y) = x^4 + y^4 - 4xy + 2$

$\nabla f(x,y) = \langle 4x^3 - 4y,\ 4y^3 - 4x \rangle$

If $\nabla f(x,y) = \vec{0}$, then

$4x^3 - 4y = 0 \Rightarrow x^3 = y$ (1)

$4y^3 - 4x = 0 \Rightarrow y^3 = x$ (2)

(1) in (2)

$(x^3)^3 = x$

$x^9 - x = 0$

$x(x^8 - 1) = 0 \Rightarrow x = 0$ or $x = \pm 1$

$\therefore$ The critical points of $f(x,y)$ are $(0,0)$, $(1,1)$ and $(-1,-1)$

To classify these points, we find the Hessian of $f(x,y)$:

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

At $(0,0)$:

$$Hf(0,0) = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \text{ and } \det(Hf(0,0)) = -16$$

Since $-16 < 0$, we conclude that at $(0,0)$, the function $f(x,y)$ has a saddle point.

At $(-1,-1)$:

$$Hf(-1,-1) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \text{ and } \det(Hf(-1,-1)) = 144 - 16 = 128.$$

Since $128 > 0$ and $f_{xx} > 0$, we conclude that the function $f(x,y)$ has a local minimum at $(-1,-1)$.

At $(1,1)$, $\det(Hf(1,1)) = 144 - 16 = 128 > 0$ and $f_{xx} > 0$, so $f(x,y)$ has a local minimum at $(1,1)$.

2. $f(x,y) = (1+xy)(x+y)$

$$\nabla f(x,y) = \langle y(x+y) + (1+xy)(1),\ x(x+y) + (1+xy)(1) \rangle$$

$$= \langle xy + y^2 + 1 + xy,\ x^2 + xy + 1 + xy \rangle$$

$$= \langle y^2 + 2xy + 1,\ x^2 + 2xy + 1 \rangle$$

If $\nabla f(x,y) = \vec{0}$, then

$$y^2 + 2xy + 1 = 0 \quad (1)$$

$$x^2 + 2xy + 1 = 0 \quad (2)$$

From (1):

$$\frac{y^2+1}{-2y} = x$$

in (2)

$$\left(\frac{y^2+1}{-2y}\right)^2 + 2\left(\frac{y^2+1}{-2y}\right) y + 1 = 0$$

$$\frac{(y^2+1)^2}{4y^2} - (y^2+1) + 1 = 0$$

(Multiplying by $4y^2$)

$$(y^2+1)^2 - 4y^2(y^2+1) + 4y^2 = 0$$

$$y^4 + 2y^2 + 1 - 4y^2(y^2+1) + 4y^2 = 0$$

$$y^4 + 2y^2 + 1 - 4y^4 - \cancel{4y^2} + \cancel{4y^2} = 0$$

$$-3y^4 + 2y^2 + 1 = 0$$

If $a = y^2$, then

$$-3a^2 + 2a + 1 = 0$$

$$\Rightarrow a = \frac{-2 \pm \sqrt{(2)^2 - 4(-3)(1)}}{2(-3)}$$

$$= \frac{-2 \pm \sqrt{4+12}}{-6} = \frac{-2 \pm \sqrt{16}}{-6} = \frac{-2 \pm 4}{-6}$$

$$a_1 = \frac{2}{-6} = -\frac{1}{3} \quad ; \quad a_2 = \frac{-6}{-6} = 1$$

For $a = -\frac{1}{3}$, there is no real solution for y.

For $a = 1$, $y^2 = 1 \Rightarrow y = \pm 1$

Therefore

$$\frac{y^2+1}{-2y} = x \Rightarrow x = \frac{1+1}{-2} = \frac{2}{-2} = -1$$

$$x = \frac{1+1}{2} = \frac{2}{2} = 1$$

So $(1,-1)$ and $(-1,1)$ are critical points of $f(x,y)$

The Hessian of $f(x,y)$ is

$$Hf(x,y) = \begin{pmatrix} 2y & 2x+2y \\ 2x+2y & 2x \end{pmatrix}$$

At $(1,-1)$:

$$Hf(1,-1) = \begin{pmatrix} -2 & 2-2 \\ 2-2 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

and $\det(Hf(1,-1)) = -4 < 0$, so at $(1,-1)$ $f(x,y)$ has a saddle point.

At $(-1,1)$:

$$Hf(-1,1) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \det(Hf(-1,1)) = -4 < 0$$

so at $(-1,1)$ $f(x,y)$ also has a saddle point.

3. $f(x,y) = y^2 - 2y\cos x$, $1 \le x \le 7$

$\nabla f(x,y) = \langle 2y\sin x, 2y - 2\cos x \rangle$

If $\nabla f(x,y) = \vec{0}$, then

$2y\sin x = 0$ ①

$2y - 2\cos x = 0$ ②

① is satisfied if $y=0$ or $\sin x = 0$, which occurs when $x = \pi$ and $x = 2\pi$ (in $1 \le x \le 7$).

From ② we get $y = \cos x$ and since $y=0$ (from ①) we have that also $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ satisfy the system of equations.

So, the critical points when $1 \le x \le 7$ are

$\left(\frac{\pi}{2}, 0\right)$, $(\pi, -1)$, $\left(\frac{3\pi}{2}, 0\right)$ and $(2\pi, 1)$

$$Hf(x,y) = \begin{pmatrix} 2y\cos x & 2\sin x \\ 2\sin x & 2 \end{pmatrix}$$

$$Hf\left(\tfrac{\pi}{2},0\right) = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}, \text{ so } \det\left(Hf\left(\tfrac{\pi}{2},0\right)\right) = -4$$

So at $\left(\frac{\pi}{2},0\right)$ $f(x,y)$ has a saddle point.

$$Hf(\pi,-1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \text{ so } \det\left(Hf(\pi,-1)\right) = 4$$

Since $f_{xx} > 0$, then at $(\pi,-1)$ $f(x,y)$ has a local minimum.

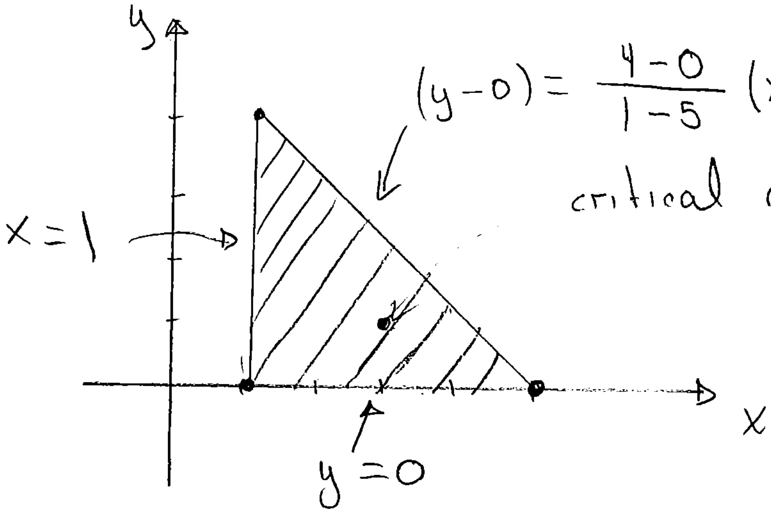
$$Hf\left(\tfrac{3\pi}{2},0\right) = \begin{pmatrix} 0 & -2 \\ -2 & 2 \end{pmatrix}, \det\left(Hf\left(\tfrac{3\pi}{2},0\right)\right) = -4$$

So at $\left(\frac{3\pi}{2},0\right)$ $f(x,y)$ has a saddle point.

$$Hf(2\pi,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \det\left(Hf(2\pi,1)\right) = 4$$

Since $f_{xx} > 0$, at $(2\pi,1)$ $f(x,y)$ has a local minimum.

4. $f(x,y) = 3 + xy - x - 2y$, on region



$(y-0) = \frac{4-0}{1-5}(x-5) = -\frac{4}{4}(x-5) = -x+5$

critical point.

$\nabla f(x,y) = \langle y-1,\ x-2 \rangle$

If $\nabla f(x,y) = \vec{0}$, then

$y = 1$ and $x = 2$

$$Hf(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det(Hf(2,1)) = -1 < 0$$

So at $(2,1)$ the function has a saddle point.

Along $x = 1$

$f(x=1, y) = 3 + y - 1 - 2y = 2 - y = u(y)$

$\Rightarrow u'(y) = -1 \therefore$ there is no local min/max along $x = 1$

Along $y = 0$

$f(x, y=0) = 3 - x \Rightarrow$ again, no local min/max along $y = 0$

Along $y = 5 - x$

$$f(x, 5-x) = 3 + x(5-x) - x - 2(5-x)$$
$$= 3 + 5x - x^2 - x - 10 + 2x$$
$$= -7 + 6x - x^2 = v(x)$$

$\Rightarrow v'(x) = 6 - 2x = 0 \Rightarrow x = 3$

so at $(3, 2)$ $f(x,y)$ has a local maximum along $y = 5 - x$.

Comparing values of $f(x,y)$ at vertices of region critical points and local minima / maxima along boundaries:

$$f(1,0) = 3 - 1 = 2$$
$$f(5,0) = 3 - 5 = -2$$
$$f(1,4) = 3 + 4 - 1 - 8 = 7 - 9 = -2$$
$$f(2,1) = 3 + 2 - 2 - 2 = 5 - 4 = 1$$
$$f(3,2) = \cancel{3} + 6 - \cancel{3} - 4 = 2$$

$\therefore$ in region, at $(1,0)$ & $(3,2)$ f has absolute maxima $f(1,0) = f(3,2) = 2$ and at $(5,0)$ and $(1,4)$ $f(x,y)$ has absolute minima $f(5,0) = f(1,4) = -2$

5. $d^2 = (x-2)^2 + (y-1)^2 + (z+1)^2 = f(x,y)$

So $\nabla f(x,y,z) = \langle 2(x-2), 2(y-1), 2(z+1) \rangle$

and $g(x,y,z) = x + y - z - 1 = 0$

So $\nabla g(x,y,z) = \langle 1, 1, -1 \rangle$

Using Lagrange multipliers:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

So

$$2(x-2) = \lambda \Rightarrow x = \frac{\lambda+4}{2} \quad (1)$$
$$2(y-1) = \lambda \Rightarrow y = \frac{\lambda+2}{2} \quad (2)$$
$$2(z+1) = -\lambda \Rightarrow z = \frac{-\lambda-2}{2} \quad (3)$$
$$x + y - z - 1 = 0 \quad (4)$$

(1), (2) and (3) in (4)

$$\frac{\lambda+4}{2} + \frac{\lambda+2}{2} - \left(\frac{-\lambda-2}{2}\right) - 1 = 0$$

$$\lambda + 3 + \frac{\lambda + \cancel{2}}{\cancel{2}} \cancel{-1} = 0$$

$$\frac{3}{2}\lambda + 3 = 0 \Rightarrow \lambda = \frac{2}{3}(-3) = -2$$

So $x = \frac{-2+4}{2} = 1$

$y = \frac{-2+2}{2} = 0$

$z = \frac{2-2}{2} = 0$

and $d^2 = (1-2)^2 + (0-1)^2 + (0+1)^2$

$= 1 + 1 + 1 = 3$

$\Rightarrow$ $\underline{d = \sqrt{3}}$

Using vectors:

The normal vector of the plane is

$\vec{n} = \langle 1, 1, -1 \rangle$

A point on the plane is $(0,0,-1)$, so

a vector from $(0,0,-1)$ to $(2,1,-1)$ is

$\vec{v} = \langle 2-0,\ 1-0,\ -1-(-1) \rangle = \langle 2, 1, 0 \rangle$

$$d = \frac{\vec{n} \cdot \vec{v}}{\|\vec{n}\|} = \frac{2+1+0}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \underline{\sqrt{3}}$$

6. $d^2 = f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$

$$g(x,y,z) = 3x + y - z - 2 = 0$$

$$h(x,y,z) = x - y + z - 2 = 0$$

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$

$$\langle 2(x-1), 2(y-1), 2(z-1) \rangle = \lambda \langle 3, 1, -1 \rangle + \mu \langle 1, -1, 1 \rangle$$

$\Rightarrow$

$$2(x-1) = 3\lambda + \mu \Rightarrow x = \frac{3\lambda+\mu}{2} + 1 \quad (1)$$

$$2(y-1) = \lambda - \mu \Rightarrow y = \frac{\lambda-\mu}{2} + 1 \quad (2)$$

$$2(z-1) = -\lambda + \mu \Rightarrow z = \frac{-\lambda+\mu}{2} + 1 \quad (3)$$

$$3x + y - z - 2 = 0 \quad (4)$$

$$x - y + z - 2 = 0 \quad (5)$$

(1), (2), (3) in (4) & (5)

$$3\left(\frac{3\lambda+\mu}{2} + 1\right) + \left(\frac{\lambda-\mu}{2} + 1\right) - \left(\frac{-\lambda+\mu}{2} + 1\right) - 2 = 0$$

$$\frac{9\lambda+3\mu}{2} + 3 + \frac{\lambda-\mu}{2} + \cancel{1} + \frac{\lambda-\mu}{2} - \cancel{1} - 2 = 0$$

$$9\lambda + 3\mu + 6 + 2\lambda - 2\mu - 4 = 0$$

$$11\lambda + \mu + 2 = 0 \quad (6)$$

$$\left(\frac{3\lambda+\mu}{2}+1\right)-\left(\frac{\lambda-\mu}{2}+1\right)+\left(\frac{-\lambda+\mu}{2}+1\right)-2=0$$

$$\frac{3\lambda+\mu}{2}+1-\frac{(\lambda-\mu)}{2}-\cancel{1}-\frac{(\lambda-\mu)}{2}+\cancel{1}-2=0$$

$$3\lambda+\mu+2-2\lambda+2\mu-4=0$$

$$\lambda+3\mu-2=0 \quad (7)$$

Solving (6) and (7):

$$11\lambda+\mu+2=0$$

$$\lambda+3\mu-2=0 \Rightarrow \lambda=2-3\mu$$

$$11(2-3\mu)+\mu+2=0$$

$$22-33\mu+\mu+2=0$$

$$24-32\mu=0 \Rightarrow \mu=\frac{24}{32}=\frac{3}{4}$$

$$\therefore \lambda=2-\frac{9}{4}=\frac{-1}{4}$$

So $x=\dfrac{3\left(\frac{-1}{4}\right)+\frac{3}{4}}{2}+1=1$

$y=\dfrac{-\frac{1}{4}-\frac{3}{4}}{2}+1=\dfrac{1}{2}$

$$z = \frac{-\left(-\frac{1}{4}\right)+\frac{3}{4}}{2} + 1 = \frac{3}{2}$$

So, the closest point to $(1,1,1)$ on the line of intersection of the given planes is

$$\underline{\left(1, \frac{1}{2}, \frac{3}{2}\right)}$$

7. $f(x,y) = 4x + 6y \qquad g(x,y) = xy - 1 = 0$

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

$$\langle 4, 6 \rangle = \lambda \langle y, x \rangle$$

$$\left.\begin{array}{ll} (1) & 4 = \lambda y \\ (2) & 6 = \lambda x \\ (3) & xy - 1 = 0 \end{array}\right\}$$

From (1): $y = \frac{4}{\lambda}$; From (2): $x = \frac{6}{\lambda}$

In (3):

$$\left(\frac{6}{\lambda}\right)\left(\frac{4}{\lambda}\right) - 1 = 0$$

$$\frac{24}{\lambda^2} = 1 \Rightarrow \lambda^2 = 24 \Rightarrow \lambda = \pm\sqrt{24}$$
$$= \pm 2\sqrt{6}$$

So $x = \dfrac{6}{\pm 2\sqrt{6}} = \pm \dfrac{3}{\sqrt{6}}$

$y = \dfrac{4}{\pm 2\sqrt{6}} = \pm \dfrac{2}{\sqrt{6}}$

$f\left(\frac{3}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) = 4\left(\frac{3}{\sqrt{6}}\right) + 6\left(\frac{2}{\sqrt{6}}\right) = \frac{24}{\sqrt{6}} = 4\sqrt{6}$ MAXIMUM

$f\left(-\frac{3}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) = -\frac{12}{\sqrt{6}} - \frac{12}{\sqrt{6}} = \frac{-24}{\sqrt{6}} = -4\sqrt{6}$ MINIMUM

However, consider what happens when $x \to 0$ or $y \to 0$. In those cases it is indeed possible to find values of $f(x,y)$ that are greater (or less) than $4\sqrt{6}$ (or $-4\sqrt{6}$). So in fact, in this example, the method of Lagrange multipliers fails to detect the extreme values of $f(x,y)$ subject to $xy=1$.

8. $f(x,y,z) = xyz \qquad g(x,y,z) = x^2+2y^2+3z^2-6=0$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$\langle yz, xz, xy\rangle = \lambda\langle 2x, 4y, 6z\rangle$

$yz = 2\lambda x \Rightarrow xyz = 2\lambda x^2$ (mutiplying by x)

$xz = 4\lambda y \Rightarrow xyz = 4\lambda y^2$ (" by y)

$xy = 6\lambda z \Rightarrow xyz = 6\lambda z^2$ (" by z)

So

$2\lambda x^2 = 4\lambda y^2 = 6\lambda z^2$ (dividing by 2λ) (assuming $\lambda \neq 0$)

$x^2 = 2y^2 = 3z^2$ (1)

(1) in $g(x,y,z)$:

$3x^2-6=0 \Rightarrow x^2=2 \Rightarrow x=\pm\sqrt{2}$

$6y^2-6=0 \Rightarrow y^2=1 \Rightarrow y=\pm 1$

$9z^2-6=0 \Rightarrow z^2=\frac{2}{3} \Rightarrow z=\pm\sqrt{\frac{2}{3}}$

So, the candidate points are

a) $(\sqrt{2}, 1, \sqrt{\frac{2}{3}})$ c) $(\sqrt{2}, -1, \sqrt{\frac{2}{3}})$ e) $(-\sqrt{2}, 1, \sqrt{\frac{2}{3}})$

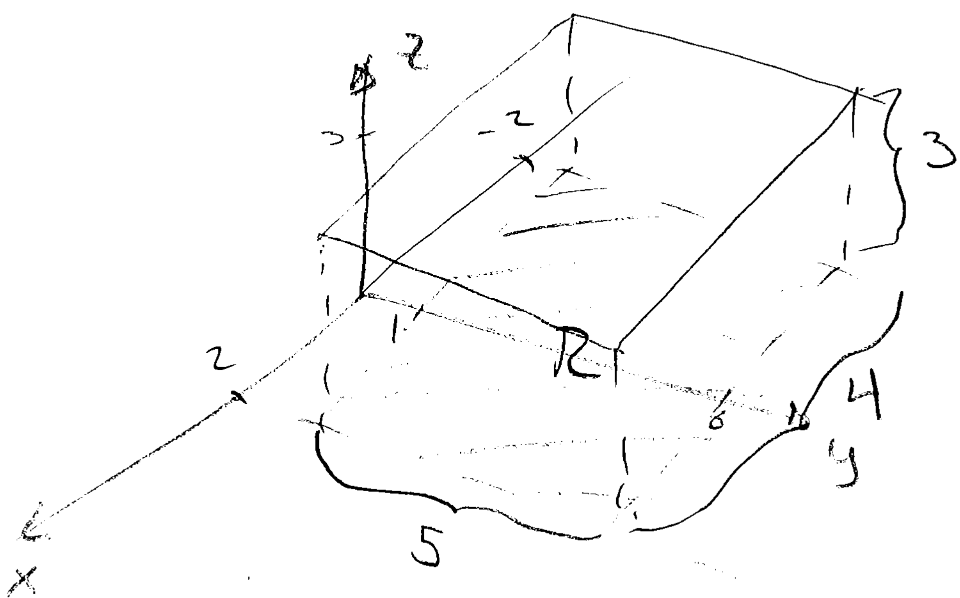
b) $(\sqrt{2}, 1, -\sqrt{\frac{2}{3}})$ d) $(\sqrt{2}, -1, -\sqrt{\frac{2}{3}})$ f) $(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}})$

and $g)\left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$ $h)\left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$

At points a, d, f and g, the function has maxima $\boxed{f(x,y,z) = \frac{2}{\sqrt{3}}}$

At b, c, e and h, The function has minima $\boxed{f(x,y,z) = -\frac{2}{\sqrt{3}}}$

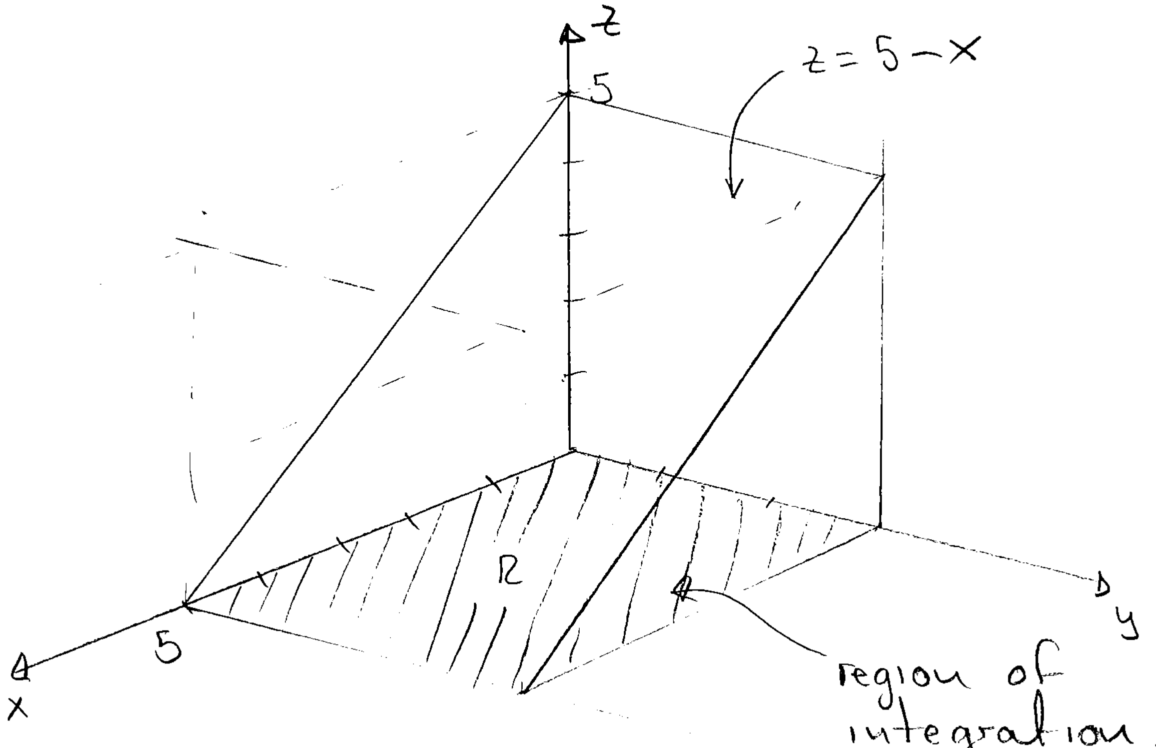
9.



So $\displaystyle\iint_R 3\, dA = (4)(5)(3) = 60$

$R = \{(x,y) \mid -2 \le x \le 2,\ 1 \le y \le 6\}$

10. A sketch of the sought volume is given below



Based on the geometric interpretation of a double integral, we have

$$\iint_R (5-x)\,dA = \underbrace{(3)(5)(5)}_{\text{box volume}} \cdot \underbrace{\left(\frac{1}{2}\right)}_{\substack{\text{we}\\ \text{need}\\ \text{just}\\ \text{half of}\\ \text{full box}}} = \frac{75}{2} = 37.5$$

11.

$$\iint_R (5-x)\, dA = \int_0^5\int_0^3 (5-x)\, dy\, dx = \int_0^5 5y - xy\Big|_0^3\, dx =$$

$$\int_0^5 (15-3x)\, dx = 15x - \frac{3}{2}x^2\Big]_0^5 = 75 - \frac{3}{2}(25) = 75 - \frac{75}{2}$$

$$= \frac{75}{2} = \underline{37.5}$$

12.

$$\int_0^1\int_1^2 (4x^3 - 9x^2y^2)\, dy\, dx = \int_0^1 \left(4x^3y - 9x^2\frac{y^3}{3}\right)\Big|_1^2\, dx =$$

$$\int_0^1 \left[(8x^3 - 24x^2) - (4x^3 - 3x^2)\right] dx =$$

$$\int_0^1 (4x^3 - 21x^2)\, dx = x^4 - 7x^3\Big]_0^1 = \underline{-6}$$

13.

$$\int_0^1\int_1^2 \frac{xe^x}{y}\, dy\, dx = \int_0^1 xe^x \ln(y)\Big|_1^2\, dx$$

$$= \ln(2)\int_0^1 xe^x\, dx = \left[\begin{matrix}\text{By parts: } u = x & v' = e^x \\ u' = 1 & v = e^x\end{matrix}\right]$$

$$= \ln(2)\left[xe^x\Big|_0^1 - \int_0^1 e^x\, dx\right] = \ln(2)\left[e - e^x\Big|_0^1\right] = \underline{\ln(2)}$$

14.

$$\iint_R \cos(x+2y)\,dA \quad , \quad R=\{(x,y) \mid 0 \le x \le \pi ,\ 0 \le y \le \pi/2\}$$

$$\iint_R \cos(x+2y)\,dA = \int_0^{\pi}\int_0^{\pi/2} \cos(x+2y)\,dy\,dx = \int_0^{\pi} \frac{1}{2}\sin(x+2y)\Big|_0^{\pi/2} dx =$$

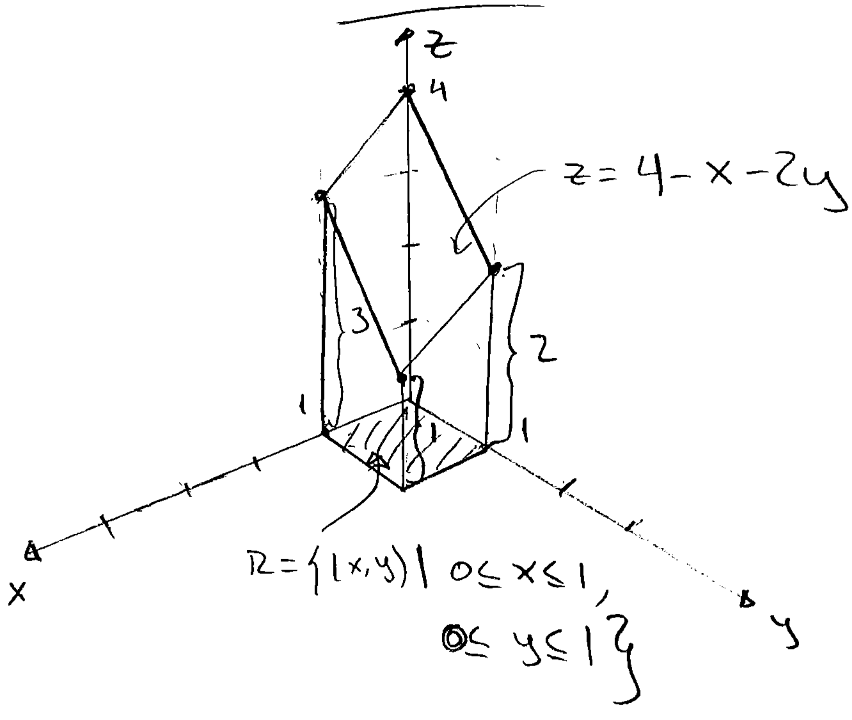
$$\frac{1}{2}\int_0^{\pi} (\sin(x+\pi) - \sin(x))\,dx \ , \ \text{but } \sin(x+\pi) = -\sin(x)$$

So

$$\frac{1}{2}\int_0^{\pi} (\sin(x+\pi) - \sin(x))\,dx = \frac{1}{2}\int_0^{\pi} -2\sin(x)\,dx = -\int_0^{\pi} \sin(x)\,dx =$$

$$\cos(x)\Big|_0^{\pi} = \cos(\pi) - 1 = \underline{-2}$$

15.



16.

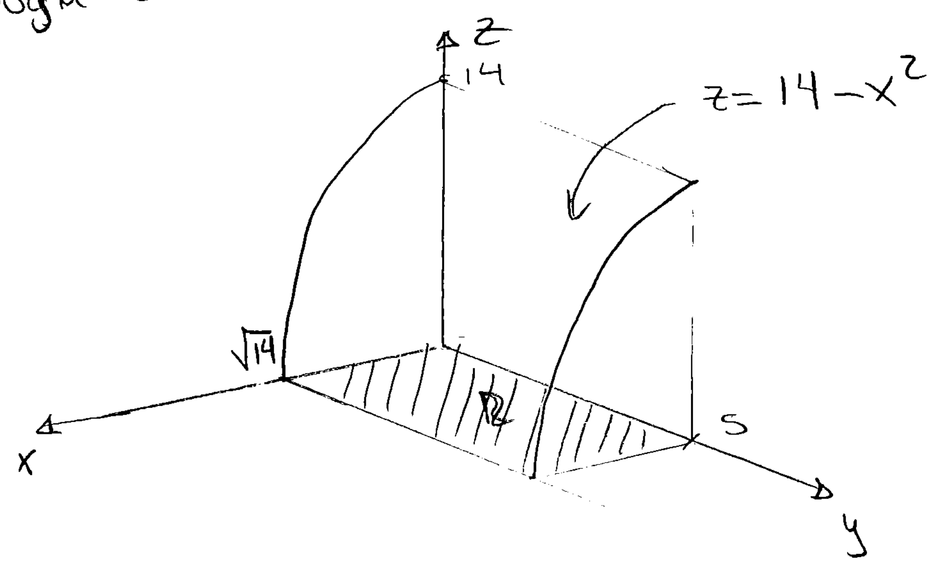
$$V=\iint_R (4+x^2-y^2)\,dA=\int_{-1}^{1}\int_0^2 (4+x^2-y^2)\,dy\,dx$$

$$=\int_{-1}^{1} 4y+x^2y-\frac{y^3}{3}\Big|_0^2\,dx=\int_{-1}^{1}\left(8+2x^2-\frac{8}{3}\right)dx=$$

$$\int_{-1}^{1}\left(\frac{16}{3}+2x^2\right)dx=\frac{16}{3}x+\frac{2}{3}x^3\Big|_{-1}^{1}=\left(\frac{16}{3}+\frac{2}{3}\right)-\left(-\frac{16}{3}-\frac{2}{3}\right)$$

$$=\left(\frac{18}{3}\right)-\left(-\frac{18}{3}\right)=6+6=\underline{12}$$

17. A rough sketch of the situation is shown below



So, the volume of this solid is given by

$$V=\iint_R (14-x^2)\,dA,\quad R=\{(x,y)\mid 0\le x\le \sqrt{14},\ 0\le y\le 5\}$$

So

$$V = \int_0^{\sqrt{14}}\int_0^5 (14 - x^2)\, dy\, dx = \int_0^{\sqrt{14}} 14y - x^2 y \Big|_0^5 \, dx =$$

$$\int_0^{\sqrt{14}} (70 - 5x^2)\, dx = 70x - \frac{5}{3}x^3 \Big|_0^{\sqrt{14}} =$$

$$70\sqrt{14} - \frac{5}{3}(14\sqrt{14}) = 70\sqrt{14} - \frac{70\sqrt{14}}{3}$$

$$= \frac{2}{3}(70\sqrt{14}) = \frac{140}{3}\sqrt{14}$$

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18. $$\int_0^2\int_y^{2y} xy\, dx\, dy = \int_0^2 \frac{x^2}{2}y \Big|_y^{2y} dy = \int_0^2 \left[\left(\frac{4y^2}{2}\right)y - \left(\frac{y^2}{2}\right)y\right] dy$$

$$= \int_0^2 \left(2y^3 - \frac{y^3}{2}\right) dy = \int_0^2 \frac{3}{2}y^3\, dy = \frac{3y^4}{8}\Big|_0^2 = \underline{6}$$

19.
$$\iint_D x^3\,dA = \int_1^e\int_0^{\ln x} x^3\,dy\,dx$$

$D=\{(x,y) \mid 1\le x\le e,\ 0\le y\le \ln x\}$

$$\int_1^e\int_0^{\ln x} x^3\,dy\,dx = \int_1^e x^3 y\Big|_0^{\ln(x)}dx = \int_1^e \ln(x)\,x^3\,dx$$

By parts:

$u=\ln(x) \qquad v'=x^3$

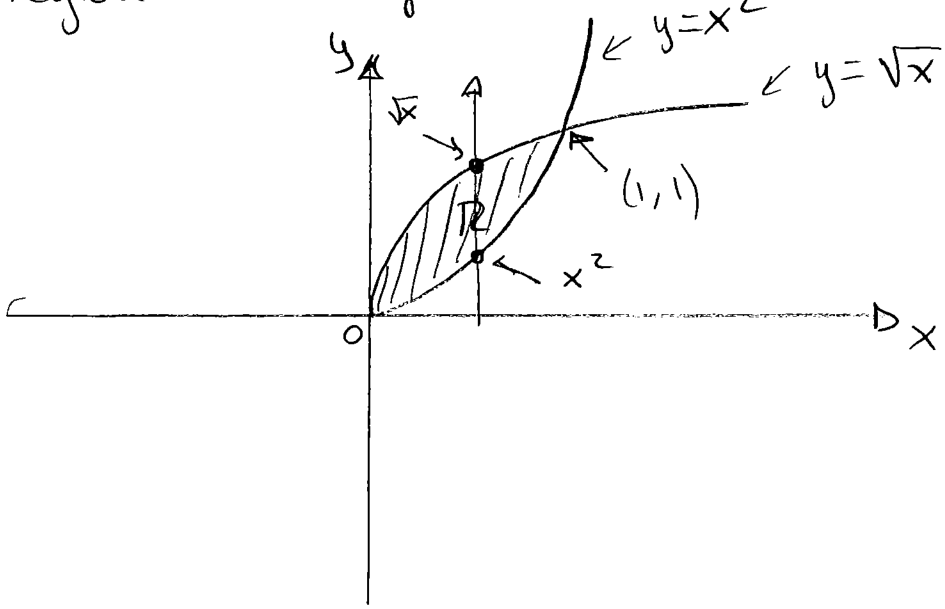
$u'=\frac{1}{x} \qquad v=\frac{x^4}{4}$

$$\int_1^e \ln(x)\,x^3 = \frac{x^4\ln(x)}{4}\Big|_1^e - \int_1^e \frac{x^3}{4}\,dx = \frac{1}{4}\left[x^4\ln(x) - \frac{x^4}{4}\right]\Big|_1^e$$

$$= \frac{1}{4}\left(e^4(1) - \frac{e^4}{4}\right) - \frac{1}{4}\left(0 - \frac{1}{4}\right)$$

$$= \frac{1}{4}\left(e^4 - \frac{e^4}{4}\right) + \frac{1}{16} = \underline{\frac{3}{16}e^4 + \frac{1}{16}}$$

20. The region of integration is shown below:



$$\iint_R (x+y)\,dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 xy + \frac{y^2}{2}\Big|_{x^2}^{\sqrt{x}}\,dx$$

$$= \int_0^1 \left[\left(x^{3/2} + \frac{x}{2}\right) - \left(x^3 + \frac{x^4}{2}\right)\right] dx$$

$$= \int_0^1 \left(x^{3/2} + \frac{x}{2} - x^3 - \frac{x^4}{2}\right) dx$$

$$= \frac{2}{5}x^{5/2} + \frac{x^2}{4} - \frac{x^4}{4} - \frac{x^5}{10}\Big|_0^1$$

$$= \frac{2}{5} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} - \frac{1}{10} = \underline{\frac{3}{10}}$$