Homework #11
Math-243 - Section 51
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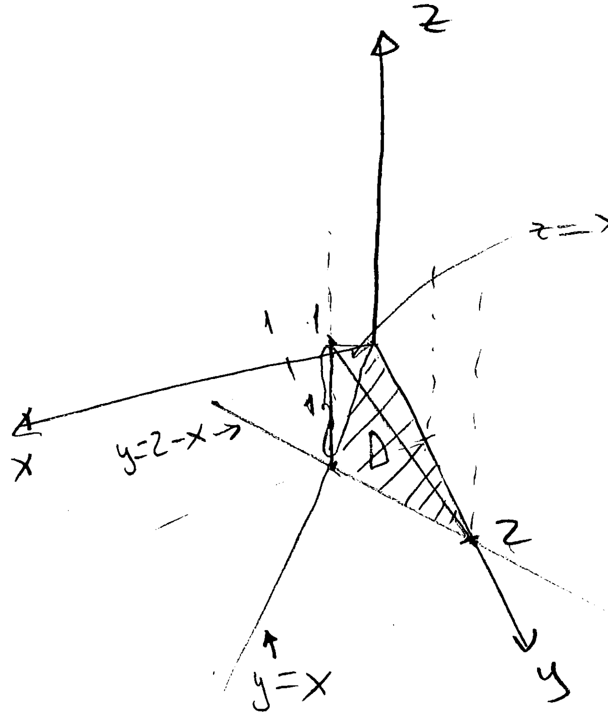
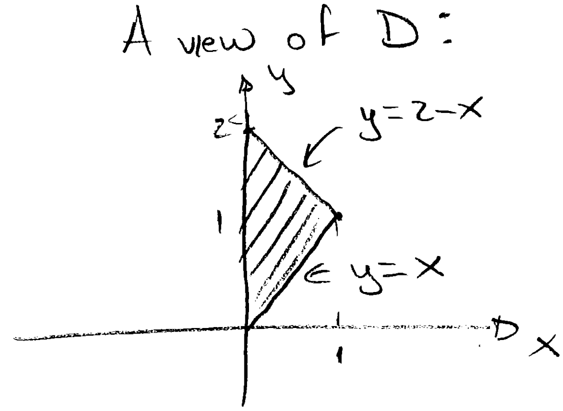
1. A sketch of the situation is shown below



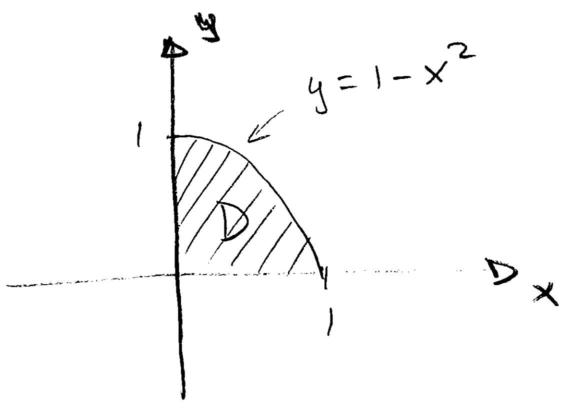


$$V=\iint_D x\,dA=\int_0^1\int_x^{2-x} x\,dy\,dx=\int_0^1 xy\Big|_x^{2-x}dx$$

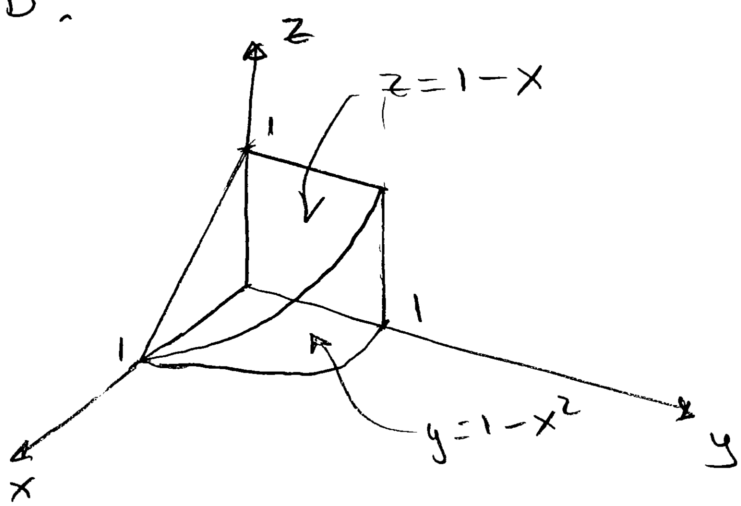
$$=\int_0^1 (2x-2x^2)\,dx=\left(x^2-\frac{2}{3}x^3\right)\Big|_0^1=\frac{1}{3}$$

2. The solid is bounded above by the plane $1-x$ and below by the region $D = \{(x,y) \mid 0 \le y \le 1-x^2,\ 0 \le x \le 1\}$

A view of D:



In 3D:



The integral is:

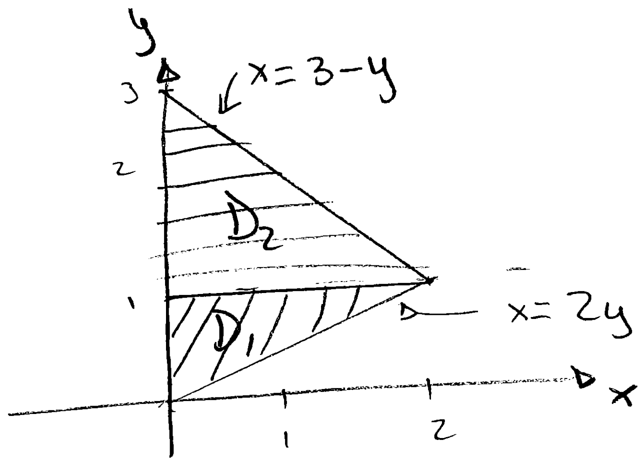
$$\int_0^1\int_0^{1-x^2} (1-x)\, dy\, dx = \int_0^1 (1-x)\, y\Big]_0^{1-x^2} dx$$

$$= \int_0^1 (1-x)(1-x^2)\,dx = \int_0^1 (1-x^2-x+x^3)\,dx$$

$$= x - \frac{x^3}{3} - \frac{x^2}{2} + \frac{x^4}{4}\Big]_0^1 = 1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12}$$

$$= \frac{5}{12}$$

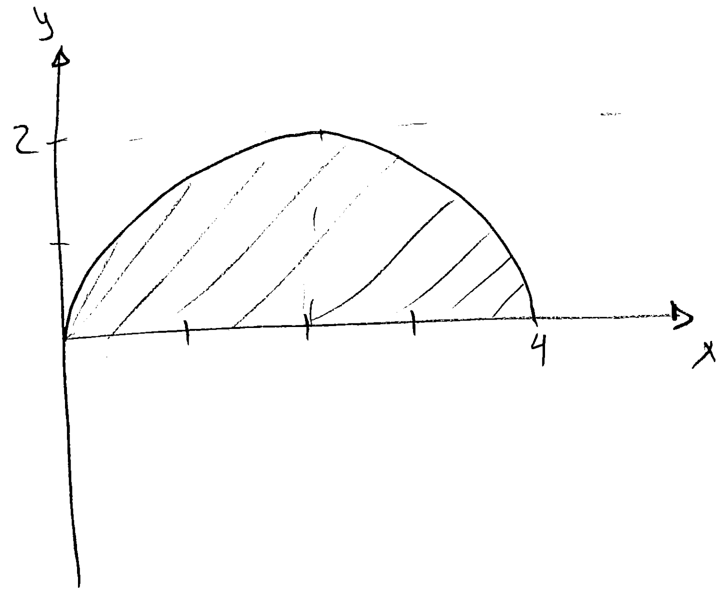
3. The regions of integration are:



So,

$$\int_0^1 \int_0^{2y} f(x,y)\,dx\,dy + \int_1^3 \int_0^{3-y} f(x,y)\,dx\,dy = \int_0^2 \int_{x/2}^{3-x} f(x,y)\,dy\,dx$$

4. A sketch of the region is below



$$\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2} \frac{r^2}{2}\Big|_0^{4\cos\theta} d\theta = \int_0^{\pi/2} \frac{16\cos^2\theta}{2}\,d\theta$$

$$= 8\int_0^{\pi/2} \cos^2\theta\,d\theta$$

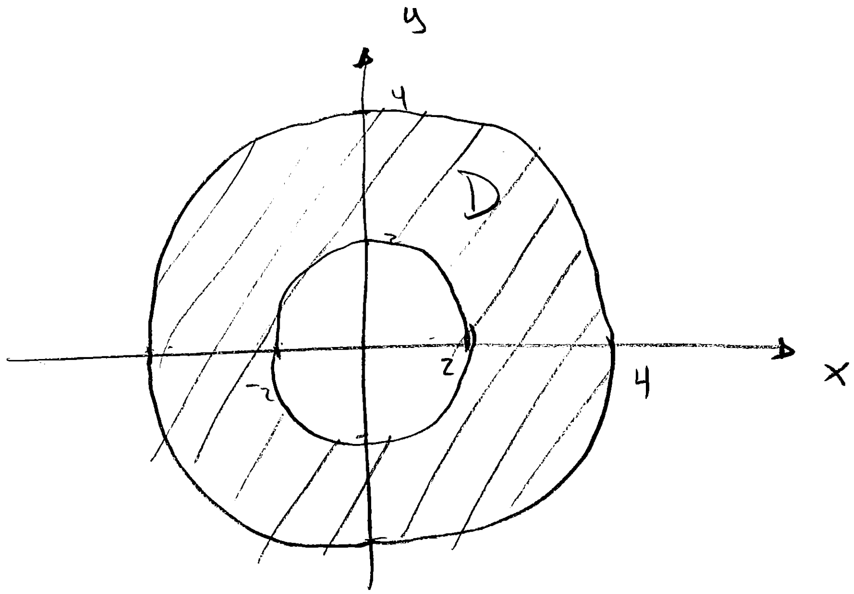
Using:

$$\frac{1+\cos(2\theta)}{2} = \cos^2\theta$$

$$= 8\int_0^{\pi/2} \frac{1}{2}\left(1+\cos(2\theta)\right)d\theta = 4\int_0^{\pi/2}\left(1+\cos(2\theta)\right)d\theta$$

$$= 4\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi/2} = 4\left(\frac{\pi}{2}+0\right) = \boxed{2\pi}$$

5. The region of integration is:



Let us denote by V the volume we are trying to find. By symetry of the sphere:

$$\frac{V}{2} = \iint_D \sqrt{16 - x^2 - y^2}\, dA$$

$$V = 2 \iint_D \sqrt{16 - x^2 - y^2}\, dA$$

Using polar coordinates:

$$V = 2 \int_0^{2\pi} \int_2^4 \sqrt{16 - r^2}\, r\, dr\, d\theta$$

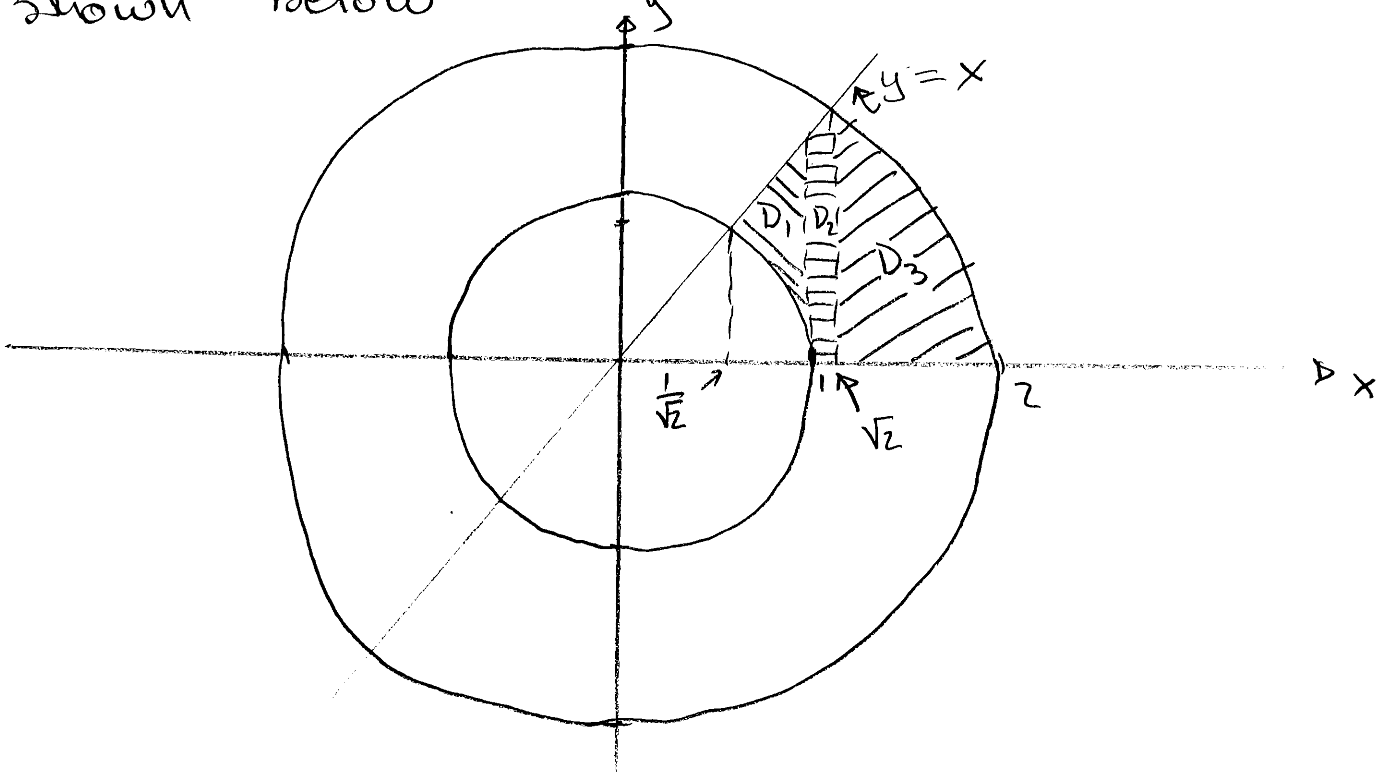
Now, if $u = 16 - r^2$, $du = -2r\,dr$, so

$$V = 2\int_0^{2\pi}\int_{12}^{0} -\frac{1}{2}\sqrt{u}\,du\,d\theta$$

$$= \int_0^{2\pi}\int_0^{12}\sqrt{u}\,du\,d\theta = \int_0^{2\pi} \frac{2}{3}u^{3/2}\Big]_0^{12} d\theta$$

$$= \frac{2}{3}\int_0^{2\pi} 8\sqrt{27}\,d\theta = \frac{16\sqrt{27}}{3}(2\pi) = \underline{32\sqrt{3}\,\pi}$$

6. A sketch of the region of integration is shown below



In polar coordinates:

$$D_1 \cup D_2 \cup D_3 = \{ (r,\theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi/4 \}$$

So

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\, dy\, dx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xy\, dy\, dx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\, dy\, dx = \int_{0}^{\pi/4}\int_{1}^{2} r^2 \cos\theta \sin\theta \; r\, dr\, d\theta$$

$$= \int_{0}^{\pi/4}\int_{1}^{2} r^3 \cos\theta \sin\theta \, dr\, d\theta = \int_{0}^{\pi/4} \frac{r^4}{4} \cos\theta \sin\theta \Big|_{1}^{2} d\theta$$

$$= \int_{0}^{\pi/4} \left(4 - \frac{1}{4}\right) \cos\theta \sin\theta \, d\theta = \frac{15}{4}\int_{0}^{\pi/4} \cos\theta \sin\theta \, d\theta$$

Using $u = \sin\theta \quad du = \cos\theta \, d\theta$

$$= \frac{15}{4}\left(\frac{\sin^2\theta}{2}\right)\Big|_{0}^{\pi/4} = \frac{15}{4}\left(\frac{\left(\frac{\sqrt{2}}{2}\right)^2}{2}\right) = \frac{15}{4}\left(\frac{1}{4}\right) = \underline{\frac{15}{16}}$$

7. $z = f(x,y) = y^2 - x^2 \Rightarrow \begin{array}{l} f_x = -2x = -2r\cos\theta \\ f_y = 2y = 2r\sin\theta \end{array}$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4r^2+1}$$

Then

$$S = \int_0^{2\pi}\int_1^2 \sqrt{4r^2+1}\; r\,dr\,d\theta \qquad \text{Using } \begin{array}{l} u = 4r^2+1 \\ du = 8r\,dr \end{array}$$

So $S = \dfrac{1}{8}\displaystyle\int_0^{2\pi}\int_5^{17} \sqrt{u}\,du\,d\theta = \dfrac{1}{8}\int_0^{2\pi} \dfrac{2}{3}u^{3/2}\Big]_5^{17} d\theta$

$$S = \frac{1}{8}\left(\frac{2}{3}\right)\left(17\sqrt{17} - 5\sqrt{5}\right)\left(2\pi\right) = \underline{\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)}$$

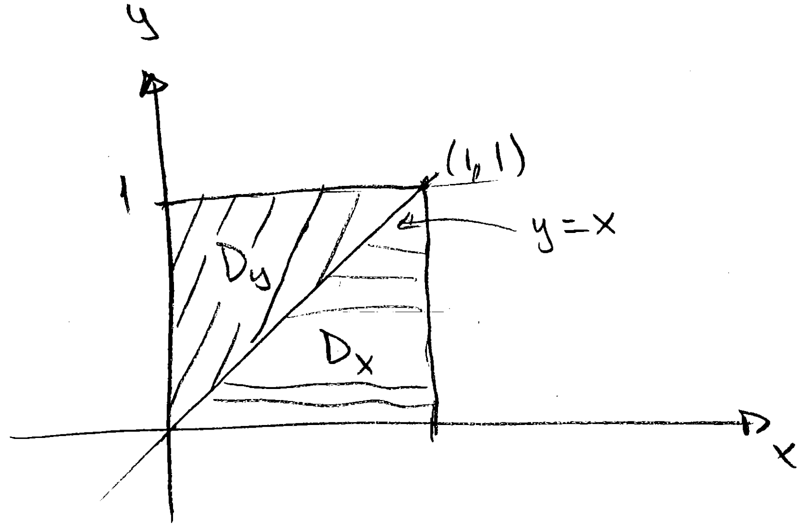
8.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \xrightarrow{\text{polar coordinates}} \lim_{a\to\infty}\int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta$$

$$\lim_{a\to\infty} \int_0^{2\pi} -\frac{1}{2} e^{-r^2}\Big|_0^a \, d\theta = -\frac{1}{2} \lim_{a\to\infty} \int_0^{2\pi} (e^{-a^2} - 1)\, d\theta$$

$$= \lim_{a\to\infty} -\frac{1}{2}\left[2e^{-a^2}\pi - 2\pi\right] = \pi - \underbrace{e^{-a^2}\pi}_{\substack{\text{as } a\to\infty \\ e^{-a^2}\to 0}} = \boxed{\pi}$$

9. The region of integration is the square:



In D_x $x > y$, so $\max\{x^2, y^2\} = x^2$

Similarly, in D_y $y > x$, so $\max\{x^2, y^2\} = y^2$

Therefore, we can split the square into D_x and D_y and integrate separately:

$$\int_0^1\int_0^1 e^{\max\{x^2, y^2\}}\, dy\, dx = \underbrace{\int_0^1\int_0^x e^{x^2}\, dy\, dx + \int_0^1\int_0^y e^{y^2}\, dx\, dy}$$

note the different orders of integration. This is done to facilitate the evaluation of the integrals

$$\int_0^1 e^{x^2} y\Big|_0^x dx + \int_0^1 e^{y^2} x\Big|_0^y dy = \int_0^1 e^{x^2} x\, dx + \int_0^1 e^{y^2} y\, dy$$

By u-subtitution:

$u = x^2$ or y^2

$du = 2x\,dx$ or $2y\,dy$

$$\frac{1}{2}\int_0^1 e^u\, du + \frac{1}{2}\int_0^1 e^u\, du = \int_0^1 e^u\, du = e^u\Big|_0^1 = \underline{e-1}$$

10.

$$\int_0^1\int_x^{2x}\int_0^y 2xyz \, dz\, dy\, dx = \int_0^1\int_x^{2x} xyz^2\Big|_0^y \, dy\, dx$$

$$= \int_0^1\int_x^{2x} xy^3 \, dy\, dx = \int_0^1 \frac{xy^4}{4}\Big|_x^{2x} dx = \frac{1}{4}\int_0^1 (16x^5 - x^5)\, dx$$

$$= \frac{1}{4}\int_0^1 15x^5\, dx = \frac{15}{4}\left(\frac{x^6}{6}\right)\Big]_0^1 = \frac{15}{(4)(6)} = \boxed{\frac{5}{8}}$$