Homework # 12

Math 243 - Section 51

Dr. Marco A. Montes de Oca

1.

$$I = \int_0^1 \int_x^{2x} \int_0^y 2xyz \; dz \, dy \, dx$$

$$= \int_0^1 \int_x^{2x} xyz^2 \Big|_0^y dy \, dx = \int_0^1 \int_x^{2x} x y^3 \, dy \, dx$$

$$= \int_0^1 \frac{xy^4}{4} \Big|_x^{2x} dx = \frac{1}{4} \int_0^1 x\left(16x^4 - x^4\right) dx$$

$$= \frac{1}{4} \int_0^1 15x^5 \, dx = \frac{15}{4} \left(\frac{x^6}{6}\right)_0^1 = \frac{15}{4(6)} = \boxed{\frac{5}{8}}$$

2.

$$I = \int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \sin y \; dy \, dz \, dx$$

$$= -\int_0^{\sqrt{\pi}} \int_0^x x^2 \cos y \Big]_0^{xz} dz \, dx$$

$$= -\int_0^{\sqrt{\pi}} \int_0^x \left(x^2 \cos(xz) - x^2\right) dz \, dx$$

$$I = -\int_0^{\sqrt{\pi}} \frac{x^2}{x}\sin(xz) - x^2 z \Big]_0^x \, dx$$

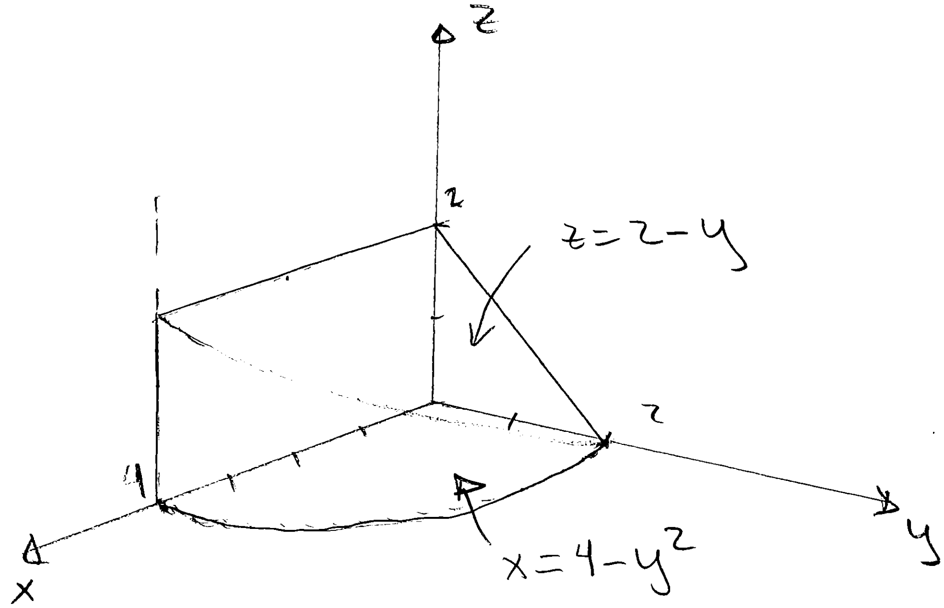
$$= -\int_0^{\sqrt{\pi}} \left(x\sin(x^2) - x^3\right) dx$$

$$= -\left[-\cos(x^2)\left(\frac{1}{2}\right) - \frac{x^4}{4}\right]_0^{\sqrt{\pi}}$$

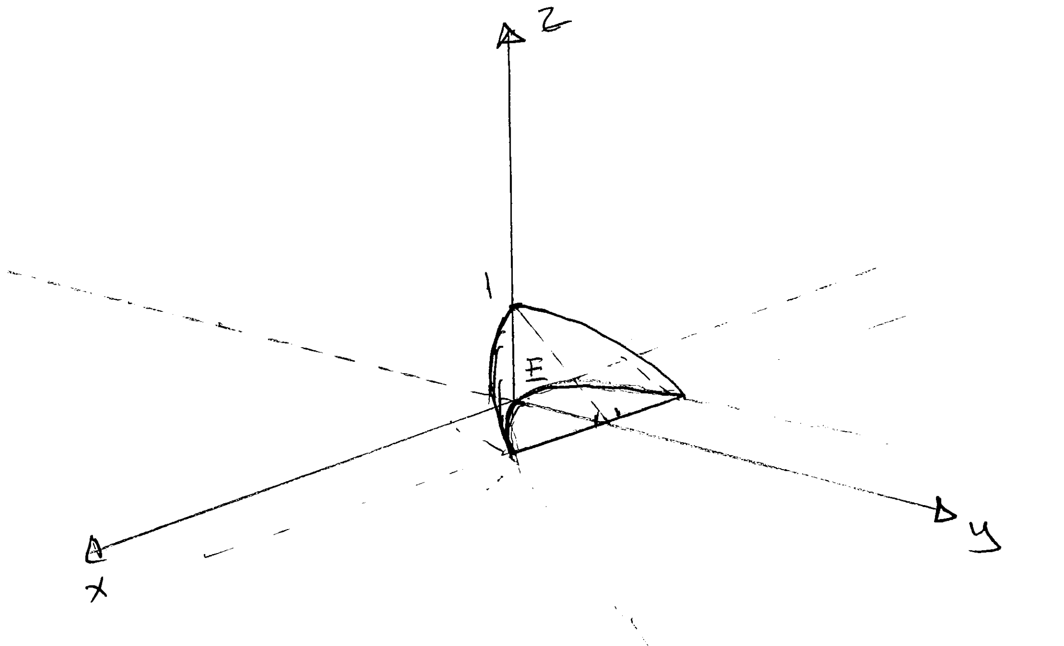
$$= -\left(\left(\frac{1}{2} - \frac{\pi^2}{4}\right) - \left(\frac{-1}{2} - 0\right)\right)$$

$$= \boxed{\frac{\pi^2}{4} - 1}$$

3.



4. A sketch of the solid is shown below



$$V = \iiint_E dV = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\, dy\, dx$$

$$V = \int_{-1}^{1}\int_{x^2}^{1} (1-y)\, dy\, dx = \int_{-1}^{1} y - \frac{y^2}{2}\Big]_{x^2}^{1} dx$$

$$= \int_{-1}^{1} \left(1 - \frac{1}{2}\right) - \left(x^2 - \frac{x^4}{2}\right) dx$$

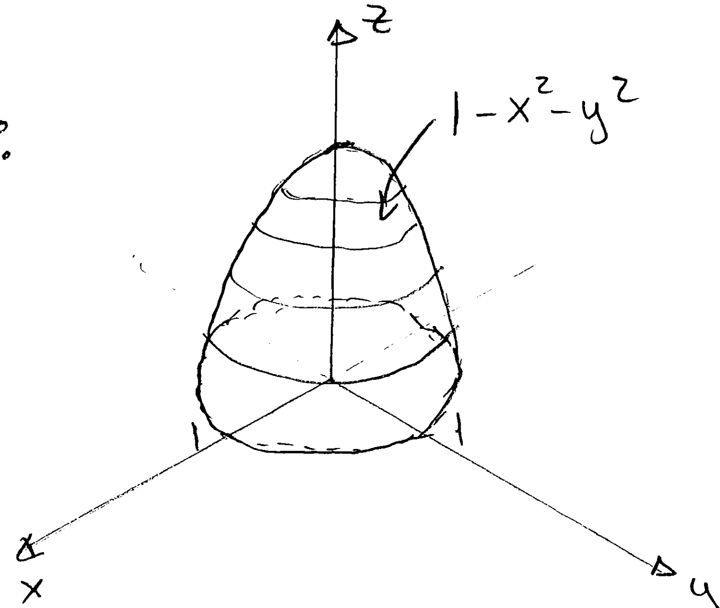
$$= \int_{-1}^{1}\left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right) dx = \frac{x}{2} - \frac{x^3}{3} + \frac{x^5}{10}\Big|_{-1}^{1}$$

$$=\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{10}\right)-\left(\frac{-1}{2}-\frac{(-1)}{3}+\frac{(-1)}{10}\right)$$

$$=1-\frac{2}{3}+\frac{2}{10}=\frac{1}{3}+\frac{1}{5}=\frac{5+3}{15}=\boxed{\frac{8}{15}}$$

5. $$V(E)=\iiint_E dV$$

E:



In cylindrical coordinates:

$$V(E)=\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1 r\underbrace{(1-r^2)}_{\substack{u=1-r^2\\du=-2r\,dr}}\,dr\,d\theta$$

$$=-\frac{1}{2}\int_0^{2\pi}\left.\frac{(1-r^2)^2}{2}\right|_0^1 d\theta=-\frac{1}{2}\int_0^{2\pi}\left(0-\frac{1}{2}\right)d\theta$$

$$=\frac{1}{4}\int_0^{2\pi}d\theta=\frac{2\pi}{4}=\frac{\pi}{2}$$

$f(x,y,z) = x^2 z + y^2 z = (x^2+y^2)z$

In cylindrical coordinates:

$(x^2+y^2)z = r^2 z$

So

$$I = \iiint_E (x^2 z + y^2 z)\, dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r^2 z \, r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left.\frac{r^3 z^2}{2}\right|_0^{1-r^2} dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{r^3(1-r^2)^2}{2}\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^3(1-2r^2+r^4)\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 (r^3 - 2r^5 + r^7)\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left.\frac{r^4}{4} - \frac{r^6}{3} + \frac{r^8}{8}\right|_0^1 d\theta$$

$$= \frac{1}{2}\left(\frac{1}{24}\right)\int_0^{2\pi} d\theta = \frac{\pi}{24}$$

$$\therefore f_{avg} = \frac{\frac{\pi}{24}}{\frac{\pi}{2}} = \boxed{\frac{1}{12}}$$

6.

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz = \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

7.

$$I = \iiint_E \sqrt{x^2+y^2}\,dV, \quad E:\{(x,y,z) \mid x^2+y^2 \le 16, -5 \le z \le 4\}$$

$$I = \int_0^{2\pi}\int_0^4\int_{-5}^4 (r)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r^2 z\Big|_{-5}^4\,dr\,d\theta$$

$$= 9\int_0^{2\pi}\int_0^4 r^2\,dr\,d\theta = 9\int_0^{2\pi} \frac{r^3}{3}\Big|_0^4\,d\theta = 3\int_0^{2\pi} 64\,d\theta = 192\int_0^{2\pi} d\theta$$

$$= \boxed{384\pi}$$

8.

$$I = \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta \cdot r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta \, z\Big|_0^{2r} dr \, d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 r^4\cos^2\theta \, dr \, d\theta = 2\int_0^{2\pi} \frac{r^5}{5}\cos^2\theta\Big|_0^1 d\theta$$

$$= \frac{2}{5}\int_0^{2\pi}\cos^2\theta \, d\theta = \frac{2}{5}\int_0^{2\pi}\frac{1+\cos(2\theta)}{2} d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}(1+\cos(2\theta)) \, d\theta = \frac{1}{5}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi}$$

$$= \frac{1}{5}\left((2\pi + 0) - (0+0)\right) = \boxed{\frac{2\pi}{5}}$$

9. In cylindrical coordinates:

Cone: $z = r$

Sphere: $z = \sqrt{2-r^2}$

$r = \sqrt{2-r^2}$

$r^2 = 2 - r^2 \Rightarrow 2r^2 = 2$

or $r = 1$

So:

$$V = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^1 \left(\sqrt{2-r^2} - r\right) r \, dr \, d\theta$$

$$V = \int_0^{2\pi} \int_0^1 \left(\sqrt{2-r^2}\, r - r^2\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{2}\left(\frac{2}{3}(2-r^2)^{3/2}\right) - \frac{r^3}{3} \right]_0^1 d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} (1+1) - (2\sqrt{2})\, d\theta$$

$$= \frac{2\sqrt{2}-2}{3} \int_0^{2\pi} d\theta = \frac{4\pi}{3}\left(\sqrt{2}-1\right)$$

10.

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz \, dz\, dx\, dy = I$$

$$I = \int_0^{2\pi} \int_0^{2} \int_r^{2} (r\cos\theta\, z)\, r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{2} r^2 \cos\theta \, z^2/2 \Big]_r^2 dr\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{2} \left(4r^2\cos\theta - r^4\cos\theta\right) dr\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{4}{3} r^3 \cos\theta - \frac{r^5}{5}\cos\theta \Big]_0^2 d\theta$$

$$I = \frac{1}{2}\int_0^{2\pi}\left(\frac{32}{3}\cos\theta - \frac{32}{5}\cos\theta\right)d\theta$$

$$= \frac{32}{15}\cancelto{0}{\int_0^{2\pi}\cos d\theta} = \boxed{0}$$