Homework #2

Math 243 - Section 51

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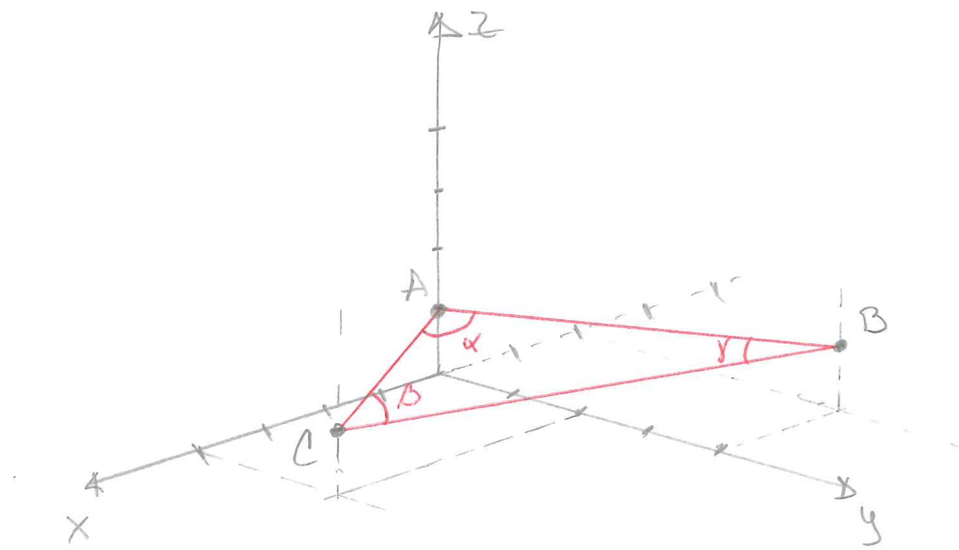
1. Written document.

2. The situation is depicted below



We want to find the values of α, β, and γ.

α: Let us define the vectors $\vec{AB}$ and $\vec{AC}$ as the vectors that represent the displacement from A to B and from A to C, respectively.

These vectors are:

$$\vec{AB} = \langle -2-0, 4-0, 1-1 \rangle = \langle -2, 4, 0 \rangle$$
$$\vec{AC} = \langle 4-0, 2-0, 1-1 \rangle = \langle 4, 2, 0 \rangle$$

α is the angle between $\vec{AB}$ and $\vec{AC}$. Thus, using the definition of dot product, we have

$$\vec{AB} \circ \vec{AC} = \|\vec{AB}\| \, \|\vec{AC}\| \cos\alpha \Rightarrow$$

$$\cos\alpha = \frac{\vec{AB} \circ \vec{AC}}{\|\vec{AB}\| \, \|\vec{AC}\|} \quad (1)$$

Now,

$$\vec{AB} \circ \vec{AC} = (-2)(4) + (4)(2) + (0)(0) = -8+8 = 0 \quad (2)$$

Substituting (2) in (1), we obtain

$\cos\alpha = 0 \Rightarrow \alpha = \frac{\pi}{2}$, that is, the vectors $\vec{AB}$ and $\vec{AC}$ are perpendicular to each other.

β: To find β, let us define the vectors $\vec{BA}$ and $\vec{BC}$ so that β is the angle between them.

$$\vec{BA} = -\vec{AB} = \langle 2, -4, 0 \rangle$$

$$\vec{BC} = \langle 4-(-2), 2-4, 1-1 \rangle = \langle 6, -2, 0 \rangle$$

$$\vec{BA} \circ \vec{BC} = \langle 2, -4, 0 \rangle \circ \langle 6, -2, 0 \rangle = 12 + 8 = 20$$

$$\|\vec{BA}\| = \sqrt{2^2 + (-4)^2 + 0^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$\|\vec{BC}\| = \sqrt{6^2 + (-2)^2 + 0^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$$

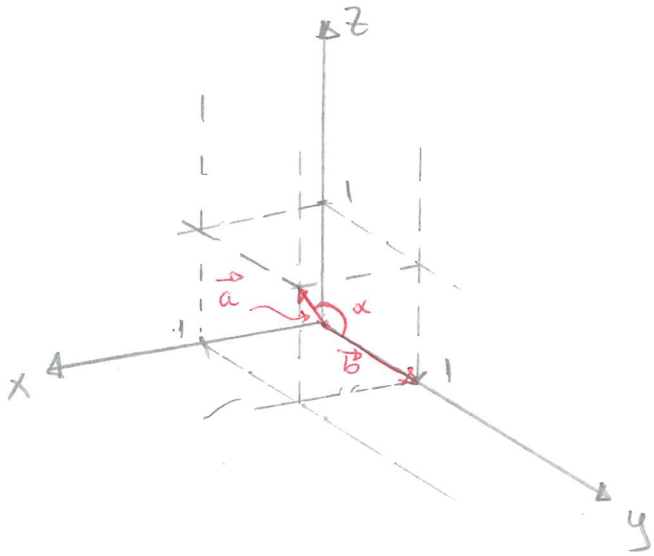
$\therefore$

$$\cos\beta = \frac{\vec{BA} \cdot \vec{BC}}{\|\vec{BA}\| \, \|\vec{BC}\|} = \frac{20}{4\sqrt{5}\sqrt{10}} = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{5}{\sqrt{2}(5)} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \beta = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

γ: Since the vertices ABC form a triangle, we can then deduce that $\gamma = \frac{\pi}{4}$ considering that the inner angles must add up π.

3. The situation is depicted below



The vector $\vec{a}$ in the figure represents one of the diagonals of the cube and we want to find the angle α, which the angle between the diagonal and one of the edges.

So, we have

$\vec{a} = \langle 1,1,1 \rangle$ and $\vec{b} = \langle 0,1,0 \rangle$

then

$$\cos\alpha = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \frac{1}{\sqrt{3}\,(1)} = \frac{1}{\sqrt{3}}$$

$\therefore$ $\underline{\alpha = 0.9553 \text{ radians or } 54.73 \text{ degrees}}$

4. The displacement vector $\vec{d} = \langle 6-1, 1-3, 10-8 \rangle = \langle 5,-2,2 \rangle$. Work is defined as $\vec{F} \cdot \vec{d}$, so

$$W = \langle 1,-6,9 \rangle \cdot \langle 5,-2,2 \rangle = (5)(1) + (-2)(-6) + (2)(9)$$

$$= 5 + 12 + 18 = \underline{35\ J}$$

5. Two vectors are perpendicular if their dot product is equal to zero. So, given $\vec{a} = \langle 2, \alpha, 1 \rangle$ and $\vec{b} = \langle 1, 3, -8 \rangle$

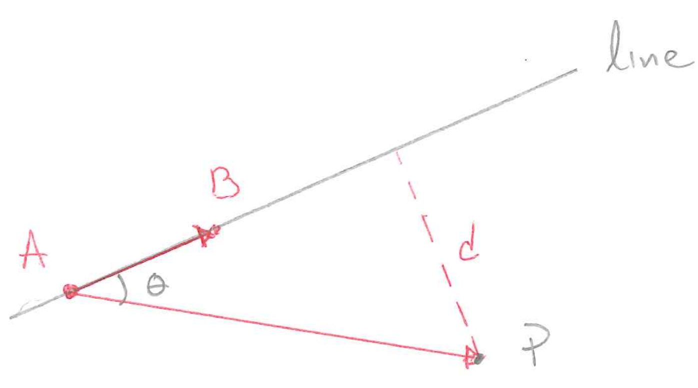
$$\vec{a} \cdot \vec{b} = (2)(1) + 3\alpha + (-8)(1) = 2 + 3\alpha - 8 = 0$$

$$3\alpha = 6$$

$$\underline{\alpha = 2}$$

6. There are several ways to tackle this problem, but in this case we are going to use vectors to do it.

Consider the following figure:



We want to find d. Note that $d = \|\vec{AP}\| \sin\theta$, and since $\|\vec{AB} \times \vec{AP}\| = \|\vec{AB}\| \, \|\vec{AP}\| \sin\theta$, then

$$d = \|\overrightarrow{AP}\| \sin\theta = \frac{\|\overrightarrow{AB} \times \overrightarrow{AP}\|}{\|\overrightarrow{AB}\|} \quad (1)$$

So, to find the distance between a point P and a line, it is enough to pick two distinct points on the line, A and B, and form vectors $\overrightarrow{AP}$ and $\overrightarrow{AB}$. Eq. (1) can then be used.

In our case, the equation of the line is $2x - 3y + 1 = 0 \Rightarrow y = \frac{2}{3}x + \frac{1}{3}$. Let point A be $A\left(0, \frac{2}{3}(0) + \frac{1}{3}\right) = A\left(0, \frac{1}{3}\right)$ and $B\left(1, \frac{2}{3}(1) + \frac{1}{3}\right) = B\left(1, 1\right)$. So $\overrightarrow{AB} = \left\langle 1-0, 1-\frac{1}{3}\right\rangle = \left\langle 1, \frac{2}{3}\right\rangle$ and $\overrightarrow{AP} = \left\langle 0-0, 0-\frac{1}{3}\right\rangle = \left\langle 0, -\frac{1}{3}\right\rangle$.

Therefore, $\overrightarrow{AB} \times \overrightarrow{AP} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & 0 \end{vmatrix} = \hat{i}\left(\frac{2}{3}(0) - \left(-\frac{1}{3}\right)0\right) - \hat{j}\left(1(0) - 0(0)\right) + \hat{k}\left(1\left(-\frac{1}{3}\right) - \frac{2}{3}(0)\right)$

$$= 0\hat{i} - 0\hat{j} - \frac{1}{3}\hat{k}$$

$$= \left\langle 0, 0, -\frac{1}{3}\right\rangle$$

$\|\vec{AB} \times \vec{AP}\| = \frac{1}{3}$ and $\|\vec{AB}\| = \sqrt{1^2 + \left(\frac{2}{3}\right)^2}$

$$= \sqrt{1 + \frac{4}{9}}$$

$$= \sqrt{\frac{13}{9}} = \frac{1}{3}\sqrt{13}$$

Therefore

$$d = \frac{\|\vec{AB} \times \vec{AP}\|}{\|\vec{AB}\|} = \frac{\frac{1}{3}}{\frac{\sqrt{13}}{3}} = \underline{\underline{\frac{1}{\sqrt{13}}}}$$

7.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -8 \\ 3 & 9 & -24 \end{vmatrix} = \hat{i}\left(3(-24) - 9(-8)\right)$$
$$-\hat{j}\left(1(-24) - 3(-8)\right)$$
$$+\hat{k}\left(1(9) - 3(3)\right) = \vec{0}$$

The cross product of two vectors is equal to $\vec{0}$ when they are parallel. In this case, $3\vec{a} = \vec{b}$.

8. The area of a triangle is half the area of the parallelogram that shares two of its sides. Then we can use the cross product to find the area of a triangle.

$\overrightarrow{AB} = \langle 0-(-1), 5-3, 2-1 \rangle = \langle 1, 2, 1 \rangle$

$\overrightarrow{AC} = \langle 4-(-1), 3-3, -1-1 \rangle = \langle 5, 0, -2 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 5 & 0 & -2 \end{vmatrix} = (2(-2) - 0(1))\hat{i} - (1(-2) - 5(1))\hat{j} + (1(0) - 5(2))\hat{k}$$

$$= \langle -4, 7, -10 \rangle$$

$$\| \overrightarrow{AB} \times \overrightarrow{AC} \| = \sqrt{(-4)^2 + 7^2 + (-10)^2} = \sqrt{16 + 49 + 100}$$

$$= \sqrt{165}$$

$\therefore$ Area of triangle $ABC = \dfrac{\sqrt{165}}{2}$

9. Let $\vec{a} = \langle 0, 1, 2 \rangle$ and $\vec{b} = \langle 1, -2, 3 \rangle$, then

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & -2 & 3 \end{vmatrix} = \left(1(3) - (-2)(2)\right)\hat{i} - \left(0(3) - (1)(2)\right)\hat{j} + \left((-2)(0) - (1)(1)\right)\hat{k}$$

$$= \langle 7, 2, -1 \rangle$$

then $$\hat{c} = \frac{\vec{c}}{\|\vec{c}\|} = \frac{1}{\sqrt{49+4+1}} \langle 7, 2, -1 \rangle = \underline{\frac{1}{\sqrt{54}} \langle 7, 2, -1 \rangle}$$

The two vectors are $\underline{\hat{c} \text{ and } -\hat{c}}$