Homework #3
Math 243 — Section 51
Dr. Marco A Montes de Oca

1. Let $\vec{v} = \langle 0-1, 2-4, 2-(-1) \rangle = \langle -1, -2, 3 \rangle$ be the direction vector of the line.

Its equation is given by:

$$\begin{aligned} \vec{r} &= \langle 1, 4, -1 \rangle + t\langle -1, -2, 3 \rangle \\ &= \langle 1-t, 4-t, -1+3t \rangle, \text{ where } t \in \mathbb{R}. \end{aligned}$$

---

2. If the line we are looking for is parallel to the line $\langle 1-2t, 3+t, -1-3t \rangle$, then they have the same direction vectors.

So if we write $\langle 1-2t, 3+t, -1-3t \rangle = \langle 1, 3, -1 \rangle + t\langle -2, 1, -3 \rangle$

then $\vec{v} = \langle -2, 1, -3 \rangle$ is the direction vector of the line we are looking for.

The vector equation of this line is:

$$\vec{r} = \langle 1, 2, 0\rangle + s\langle -2, 1, -3\rangle$$

$$= \underline{\langle 1-2s, 2+s, -3s\rangle}$$

3. If the line in question is perpendicular to the plane $x+2y-z=3$, then it is parallel to the plane's normal vector $\vec{n} = \langle 1, 2, -1\rangle$.

So, the equation of the line is

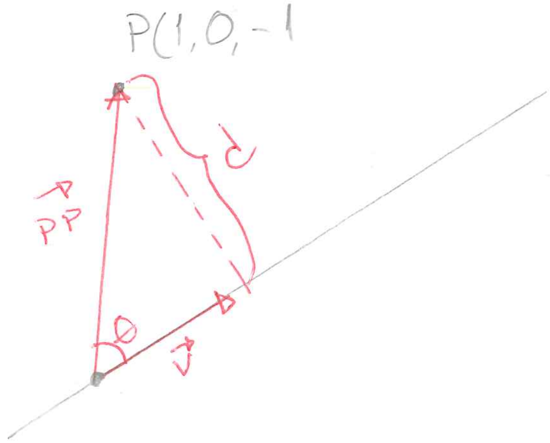
$$\vec{r} = \langle 0, 1, 1\rangle + t\langle 1, 2, -1\rangle = \underline{\langle t, 1+2t, 1-t\rangle}$$

4. Let $\vec{PP}$ be a vector from a point on the line and the point $(1, 0, -1)$. We use the reference point embedded in the line's equation

: $\vec{r} = \underbrace{\langle -1, 2, 1\rangle}_{\substack{\text{position vector} \\ \text{of } (-1,2,1).}} + t\underbrace{\langle 2, -1, 3\rangle}_{\vec{v}}$

So $\vec{PP} = \langle 1-(-1), 0-2, -1-1\rangle = \langle 2, -2, -2\rangle$

The following figure shows the situation:



We have:

$$d = \|\vec{PP}\| \sin\theta = \frac{\|\vec{PP}\|\,\|\vec{v}\| \sin\theta}{\|\vec{v}\|} = \frac{\|\vec{PP} \times \vec{v}\|}{\|\vec{v}\|}$$

$$\vec{PP} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & -2 \\ 2 & -1 & 3 \end{vmatrix} = (-6-2)\hat{i} - (6-(-4))\hat{j} + (-2-(-4))\hat{k}$$

$$= -8\hat{i} - 10 + 2\hat{k} = \langle -8, -10, 2 \rangle$$

$$\|\vec{PP} \times \vec{v}\| = \sqrt{(-8)^2 + (-10)^2 + (2)^2} = \sqrt{64 + 100 + 4} = \sqrt{168}$$

$$\|\vec{v}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{4+1+9} = \sqrt{14}$$

$$\therefore \quad d = \frac{\sqrt{168}}{\sqrt{14}} = \sqrt{\frac{168}{14}} = \underline{\sqrt{12}}$$

5. Let $A(3,-1,2)$, $B(8,2,4)$, $C(3,2,1)$.

Then
$$\vec{AB} = \langle 8-3,\ 2-(-1),\ 4-2\rangle = \langle 5,3,2\rangle$$
&
$$\vec{AC} = \langle 3-3,\ 2-(-1),\ 1-2\rangle = \langle 0,3,-1\rangle$$
are parallel to the plane. Therefore $\vec{AB}\times\vec{AC}$ is perpendicular to the plane and so

$$\vec{n} = \vec{AB}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 3 & 2 \\ 0 & 3 & -1 \end{vmatrix} = (-3-6)\hat{i} - (-5-0)\hat{j} + (15-0)\hat{k}$$
$$= -9\hat{i} + 5\hat{j} + 15\hat{k}$$

The vector equation of a plane is:

$$(\vec{r}-\vec{a})\cdot\vec{n} = 0$$

$$\Big(\langle x,y,z\rangle - \underbrace{\langle 3,-1,2\rangle}_{\substack{\text{choosing } A \\ \text{as reference} \\ \text{point}}}\Big)\cdot\langle -9,5,15\rangle = 0$$

$$\langle x-3,\ y+1,\ z-2\rangle\cdot\langle -9,5,15\rangle = 0$$

$$-9(x-3) + 5(y+1) + 15(z-2) = 0$$

$$-9x + 27 + 5y + 5 + 15z - 30 = 0$$

$$\underline{-9x + 5y + 15z + 2 = 0}$$

6. Let $\vec{A} = \langle 1, 5, -2 \rangle$, $\vec{B} = \langle 3, 1, 0 \rangle$ and $\vec{C} = \langle 5, 9, -4 \rangle$.

If $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$, then $\vec{A}$, $\vec{B}$, and $\vec{C}$ lie in the same plane.

So:

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 0 \\ 5 & 9 & -4 \end{vmatrix} = (-4-0)\hat{i} - (-12-0)\hat{j} + (27-5)\hat{k}$$
$$= -4\hat{i} + 12\hat{j} + 22\hat{k}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \langle 1, 5, -2 \rangle \cdot \langle -4, 12, 22 \rangle = -4 + 60 - 44 = 12$$

$\therefore$ $\vec{A}$, $\vec{B}$ and $\vec{C}$ are not coplanar

7. Let $A(1,3,2)$, $B(3,-1,6)$, $C(5,2,0)$ and $D(3,6,-4)$.

To know if these points lie in the same plane, we make

$$\vec{AB} = \langle 3-1, -1-3, 6-2 \rangle = \langle 2, -4, 4 \rangle$$
$$\vec{AC} = \langle 5-1, 2-3, 0-2 \rangle = \langle 4, -1, -2 \rangle$$
$$\vec{AD} = \langle 3-1, 6-3, -4-2 \rangle = \langle 2, 3, -6 \rangle$$

Then we test whether $\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = 0$.

$$\vec{AC} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = (6-(-6))\hat{i} - (-24-(-4))\hat{j} + (12-(-2))\hat{k}$$
$$= 12\hat{i} + 20\hat{j} + 14\hat{k}$$

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \langle 2, -4, 4 \rangle \cdot \langle 12, 20, 14 \rangle = 24 - 80 + 56 = \underline{0}$$

$\therefore$ Points A, B, C, D lie in the same plane.

8. If $\langle 3, 2, x \rangle$ and $\langle 2x, 4, x \rangle$ are orthogonal, then

$$\langle 3, 2, x \rangle \cdot \langle 2x, 4, x \rangle = 0$$
$$6x + 8 + x^2 = 0$$
$$(x+4)(x+2) = 0 \Rightarrow \begin{array}{c} x = -2 \\ \text{or} \\ x = -4 \end{array}$$

9. If $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$

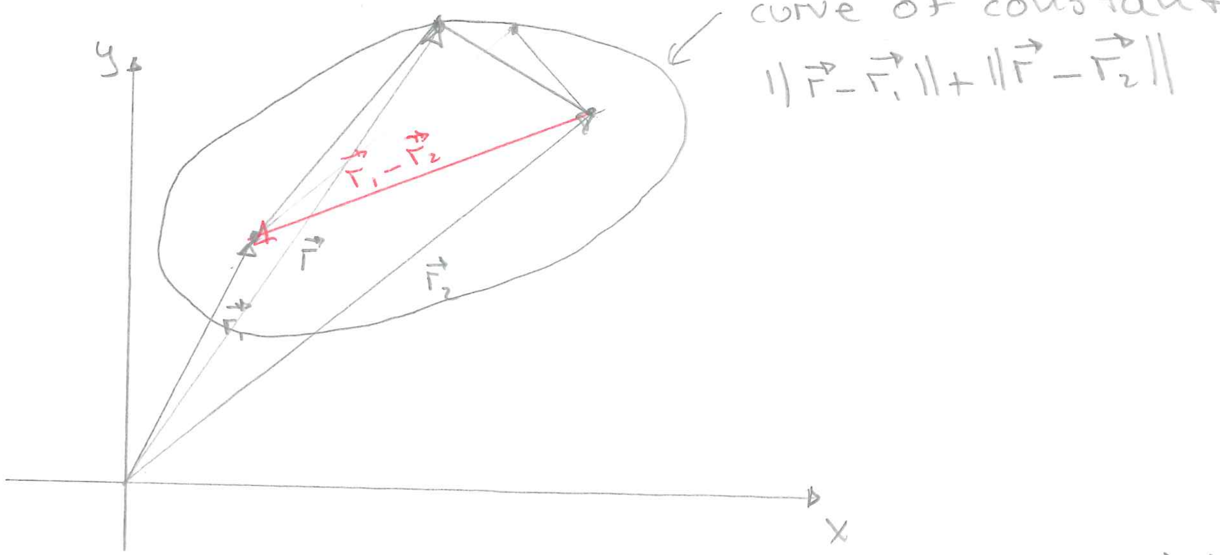
a) $(\vec{u} \times \vec{v}) \cdot \vec{w} = 2$ [because $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$]

b) $\vec{u} \cdot (\vec{w} \times \vec{v}) = -2$ [because $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (-\vec{c} \times \vec{b})$
$= -\vec{a} \cdot (\vec{c} \times \vec{b})$

c) $\vec{v} \cdot (\vec{u} \times \vec{w}) = -2$ [because $\vec{v} \cdot (\vec{u} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w} =$
$(-\vec{u} \times \vec{v}) \cdot \vec{w} = -(\vec{u} \times \vec{v}) \cdot \vec{w} =$
$-(\vec{u} \cdot (\vec{v} \times \vec{w}))$]

d) $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$ [because $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. Therefore $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$]

10. Consider the following figure:



The set of points that satisfy $\|\vec{r} - \vec{r}_1\| + \|\vec{r} - \vec{r}_2\| = k$ and $k > \|\vec{r}_1 - \vec{r}_2\|$ is an ellipse with points (x_1, y_1) and (x_2, y_2) as foci.