Homework #4

Math-243 - Section 51

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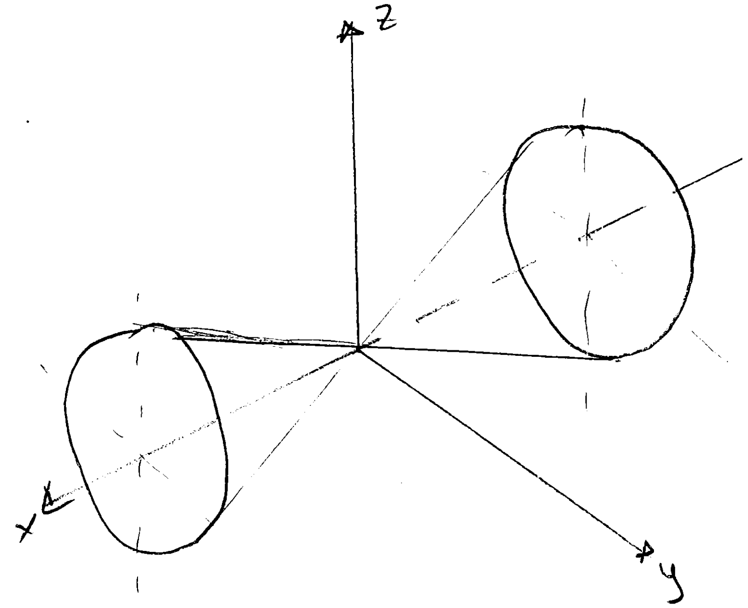
1. $x^2 = 2y^2 + 3z^2$

$\div 6$: $\dfrac{x^2}{6} = \dfrac{y^2}{3} + \dfrac{z^2}{2}$

Based on the information in Table 1, in Chapter 12, Section 12.6, this equation represents a cone parallel to the x-axis. A rough sketch of the cone is the following:

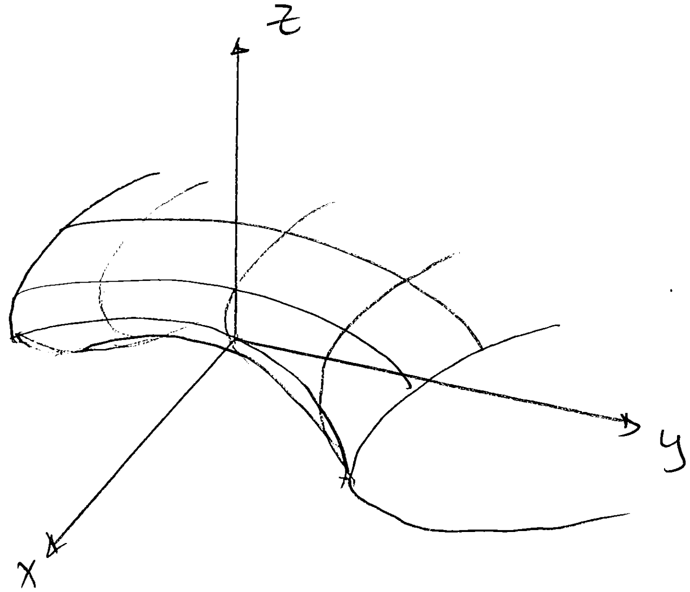


2. $4x - y^2 + 4z^2 = 0$

$4x = y^2 - 4z^2$

$x = \frac{y^2}{4} - z^2$

This equation represents a hyperbolic paraboloid parallel to the y-axis. A sketch is shown below



3. $4y^2 + z^2 - x - 16y - 4z + 20 = 0$

Rearranging terms:

$4y^2 - 16y + z^2 - 4z = x - 20$

$4(y^2 - 4y) + (z^2 - 4z) = x - 20$

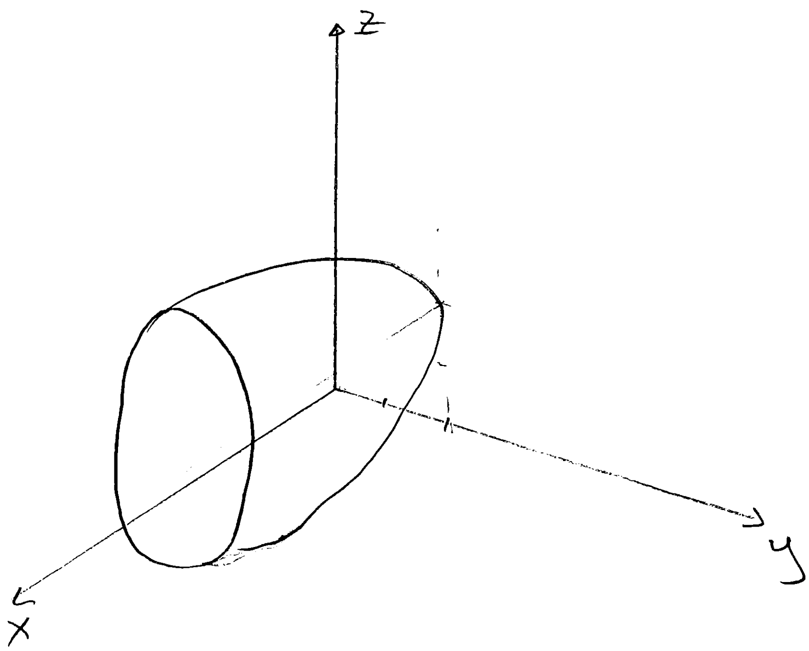
Completing squares:

$4(y^2 - 4y + 4) + (z^2 - 4z + 4) = x - 20 + 16 + 4$

$4(y-2)^2 + (z-2)^2 = x$

$$(y-2)^2 + \frac{(z-2)^2}{4} = \frac{x}{4}$$

This equation represents an elliptic paraboloid with vertex at $(0,2,2)$.



4. $x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0$

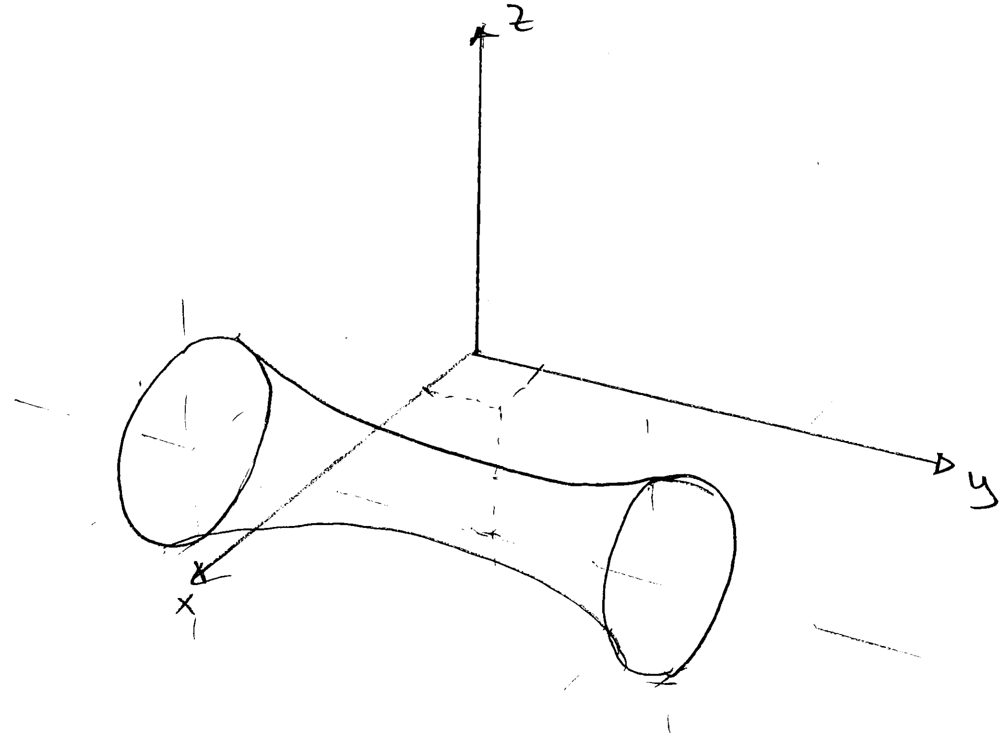
Rearranging and completing squares:

$$(x^2-2x) - (y^2-2y) + (z^2+4z) = -2$$

$$(x-1)^2 - (y-1)^2 + (z+2)^2 = -2+1-1+4 = 2$$

$$\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2} + \frac{(z+2)^2}{2} = 1$$

This is a hyperboloid with vertex at $(1,1,-2)$ parallel with the y-axis:



5. If $\vec{r}(t) = \langle t^2, 1-3t, 1+t^3 \rangle$ passes through the points $(1,4,0)$ and $(9,-8,28)$, then there must be a value of t such that:

① $t^2 = 1$
② $1-3t = 4$ in the first case
③ $1+t^3 = 0$

and

① $t^2 = 9$
② $1-3t = -8$ in the second case
③ $1+t^3 = 28$

In the first case:

From 1: $\sqrt{t^2} = \sqrt{1}$

$|t| = 1 \Rightarrow t = 1$ or $t = -1$

If $t=1$, in 2 we see

$1 - 3(1) = 4$

$-2 = 4$ which is a contradiction,

therefore t cannot be equal to 1.

If $t=-1$, then

$1 - 3(-1) = 1 + 3 = 4$

and

$1 + (-1)^3 = 1 - 1 = 0$

So, with $t=-1$, $\vec{r}(t)$ passes through $(1,4,0)$.

In the second case:

From 1: $\sqrt{t^2} = \sqrt{9}$

$|t| = 3 \Rightarrow t = 3$ or $t = -3$

Trying $t=3$ in 2:

$1 - 3(3) = 1 - 9 = -8$

$t=3$ in 3:

$1 + (3)^3 = 1 + 27 = 28$

at $t=3$, $\vec{r}(t)$ i.e. passes through $(9, -8, 28)$

If $t=-3$:

$1-3(-3)=1+9=10 \neq -8 \therefore$ at $t=-3$ $\vec{r}(t)$ cannot possibly pass through $(9,-8,28)$.

In the case of the point $(4,7,-6)$ we have:

$t^2=4 \Rightarrow |t|=2 \Rightarrow t=2$ or $t=-2$

$1-3t=7$

$1+t^3=-6$

Now, $t=2$ in $1-3t$:

$1-3(2)=1-6=-5 \neq 7 \therefore \vec{r}(t)$ at $t=2$ does not pass through $(4,7,-6)$.

When $t=-2$, we have:

$1-3(-2)=1+6=7$

and

$1+(-2)^3=1-8=-7 \neq -6 \therefore \vec{r}(t)$ does not pass through $(4,7,-6)$.

6. If $\vec{r}_1(t)$ and $\vec{r}_2(t)$ describe trajectories of two particles, they will collide if $\vec{r}_1(t) = \vec{r}_2(t)$. This implies that

① $t = 1 + 2t$
② $t^2 = 1 + 6t$
③ $t^3 = 1 + 14t$

From ①: $t - 2t = 1 \Rightarrow t = -1$.

$t = -1$ in ②: $(-1)^2 = 1 + 6(-1)$

$$\underbrace{1 = 1 - 6 = -5,}_{}$$

Contradiction $\Rightarrow$ the particles never collide.

However, the two trajectories might intersect. To see if they intersect, we have to change parameters for one trajectory, say $\vec{r}_1$ so that we allow the intersection point to be reached at different times.

So, $\vec{r}_1(s) = \vec{r}_2(t)$ means:

$s = 1 + 2t$ ① $\Rightarrow$

$s^2 = 1 + 6t$ ②

$s^3 = 1 + 14t$ ③

① in ②:

$(1+2t)^2 = 1 + 6t$

$1 + 4t + 4t^2 = 1 + 6t$

$4t^2 + 4t - 6t = 0$

$4t^2 - 2t = 0$

$t(4t-2) = 0 \Rightarrow \begin{array}{l} t = 0 \\ 4t - 2 = 0 \Rightarrow t = \frac{1}{2} \end{array}$

$\therefore$ if $t=0$, $s=1$

if $t=\frac{1}{2}$, $s=2$

$t=0$, $s=1$ in ③

$(1)^3 = 1 + 14(0) \Rightarrow 1 = 1$ ✓

$t=\frac{1}{2}$, $s=2$ in ③

$(2)^3 = 1 + 14\left(\frac{1}{2}\right)$

$8 = 1 + 7 = 8$ ✓

$\therefore$ $\vec{r}_1(s)$ and $\vec{r}_2(t)$ intersect twice:

when $t=0$, $s=1$ at $(1,1,1)$

when $t=\frac{1}{2}$, $s=2$ at $(2,4,8)$

7. $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$

$\therefore$

$$\vec{r}'(t) = \langle -2\sin(t), -2\sin(t), \sec^2(t) \rangle$$

$$\vec{r}'\left(\tfrac{\pi}{4}\right) = \left\langle -2\frac{\sqrt{2}}{2}, -2\frac{\sqrt{2}}{2}, (\sqrt{2})^2 \right\rangle$$

$$= \langle -\sqrt{2}, -\sqrt{2}, 2 \rangle$$

$$\left\|\vec{r}'\left(\tfrac{\pi}{4}\right)\right\| = \sqrt{(-\sqrt{2})^2 + (-\sqrt{2})^2 + 2^2}$$

$$= \sqrt{2+2+4} = \sqrt{8}$$

$\therefore$

$$\vec{T}\left(\tfrac{\pi}{4}\right) = \left\langle -\sqrt{\frac{2}{8}}, -\sqrt{\frac{2}{8}}, \frac{2}{\sqrt{8}} \right\rangle$$

$$= \underline{\left\langle -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2} \right\rangle}$$

8. $\vec{r}'(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t} \rangle$

$$\vec{r}'(0) = \langle 2e^0, -2e^0, (0+1)e^0 \rangle = \langle 2, -2, 1 \rangle$$

$$\|\vec{r}'(0)\| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$$

So $\vec{T}(0) = \dfrac{\vec{r}'(0)}{\|\vec{r}'(0)\|} = \underline{\left\langle \frac{2}{3}, \frac{-2}{3}, \frac{1}{3} \right\rangle}$

$\vec{r}''(t) = \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t} \rangle$

$\vec{r}''(0) = \langle 4e^0, 4e^0, (0+4)e^0 \rangle = \underline{\langle 4, 4, 4 \rangle}$

$\vec{r}'(t) \cdot \vec{r}''(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t} \rangle \cdot$
$\langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t} \rangle$

$= (2e^{2t})(4e^{2t}) + (-2e^{-2t})(4e^{-2t})$
$+ ((2t+1)e^{2t})((4t+4)e^{2t})$

$= 8e^{4t} - 8e^{-4t} + (8t^2 + 12t + 4)e^{4t}$

$= \underline{(8t^2 + 12t + 12)e^{4t} - 8e^{-4t}}$

9. If $\vec{r}(t) = \langle t, e^{-t}, 2t - t^2 \rangle$, then

$\vec{r}'(t) = \langle 1, -e^{-t}, 2 - 2t \rangle \Rightarrow$ At $(0,1,0)$, $t=0 \Rightarrow$

$\vec{r}'(0) = \langle 1, -e^0, 2-0 \rangle = \langle 1, -1, 2 \rangle$

$\vec{T}(0) = \dfrac{\vec{r}'(0)}{\|\vec{r}'(0)\|} = \dfrac{1}{\sqrt{6}} \langle 1, -1, 2 \rangle = \left\langle \dfrac{1}{\sqrt{6}}, -\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}} \right\rangle$

Thus, a vector equation of the tangent line is

$$\vec{w}(t) = \langle 0, 1, 0 \rangle + t \left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle$$

$$= \left\langle \frac{t}{\sqrt{6}}, 1 - \frac{t}{\sqrt{6}}, \frac{2t}{\sqrt{6}} \right\rangle$$

The parametric equations are:

$$x = \frac{t}{\sqrt{6}}, \quad y = 1 - \frac{t}{\sqrt{6}}, \quad z = \frac{2t}{\sqrt{6}}$$

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10. If $\vec{r}(t) = \langle t\cos(t), t, t\sin(t) \rangle$, then $\vec{r}\,'(t) = \langle \cos(t) - t\sin(t), 1, t\cos(t) + \sin(t) \rangle$.

At $(-\pi, \pi, 0)$, $t = \pi \Rightarrow$

$\vec{r}\,'(\pi) = \langle -1, 1, -\pi \rangle$. $\vec{r}\,'(\pi)$ is tangent to the curve represented by $\vec{r}(t)$ at $t = \pi$, so we can use $\vec{r}\,'(\pi)$ as the direction vector of the tangent line we are looking for.

Therefore, we get:

$$\vec{\omega}(t) = \langle -\pi, \pi, 0 \rangle + t\langle -1, 1, -\pi \rangle$$
$$= \langle -\pi - t, \pi + t, -\pi t \rangle$$

And $x = -\pi - t$

$y = \pi + t$

$z = -\pi t$