Homework #5

Math 243 – Section 51

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1. Since the angle of intersection of two curves at a point is equal to the angle between their tangent vectors at that point, we have

$$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle \quad ; \quad \vec{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$$

tangent vectors $\rightarrow$ $$\vec{r}_1{}'(t) = \langle 1, 2t, 3t^2 \rangle \quad ; \quad \vec{r}_2{}'(t) = \langle \cos t, 2\cos 2t, 1 \rangle$$

at the origin, $t=0$, so

$$\vec{r}_1{}'(0) = \langle 1, 0, 0 \rangle \quad ; \quad \vec{r}_2(0) = \langle 1, 2, 1 \rangle$$

So, since $\vec{r}_1{}'(0) \cdot \vec{r}_2(0) = \|\vec{r}_1{}'(0)\| \|\vec{r}_2(0)\| \cos\theta$

$$\cos\theta = \frac{\langle 1,0,0 \rangle \cdot \langle 1,2,1 \rangle}{(1)(\sqrt{6})} = \frac{1}{\sqrt{6}} \Rightarrow \theta = 1.1502 \text{ rad}$$

or

$65.91°$

2. If the position function of a particle is $\vec{r}(t) = e^t \langle \cos t, \sin t, t \rangle$, then:

Its velocity $\vec{v}(t) = \vec{r}\,'(t) = \left\{ \begin{array}{l} \text{Using the differentiation rule:} \\ \frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t) \end{array} \right\}$

$$\vec{r}\,'(t) = e^t \langle \cos t, \sin t, t \rangle + e^t \langle -\sin t, \cos t, 1 \rangle$$

$$= \underline{e^t \langle \cos t - \sin t, \sin t + \cos t, t+1 \rangle}$$

Its acceleration $\vec{a}(t) = \vec{r}\,''(t) = \left\{ \text{Again, using the same rule} \right\}$

$$\vec{r}\,''(t) = e^t \langle \cos t - \sin t, \sin t + \cos t, t+1 \rangle + e^t \langle -\sin t - \cos t, \cos t - \sin t, 1 \rangle$$

$$= \underline{e^t \langle -2\sin t, 2\cos t, t+2 \rangle}$$

Its speed $Sp(t) = \|\vec{r}\,'(t)\|$

$$\|\vec{r}\,'(t)\| = \sqrt{e^{2t}\left((\cos t - \sin t)^2 + (\sin t + \cos t)^2 + (t+1)^2\right)}$$

$$= e^t\sqrt{(\cos^2 t - \cancel{2\cos t \sin t} + \sin^2 t) + (\sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t) + (t^2+2t+1)}$$

$$= e^t\sqrt{1+1+t^2+2t+1} = \underline{e^t\sqrt{t^2+2t+3}}$$

3. If $\vec{r}(t) = \langle t^2, 5t, t^2 - 16t\rangle$, then $\vec{r}\,'(t) = \langle 2t, 5, 2t-16\rangle$ and speed is

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 25 + (2t-16)^2}$$

$$= \sqrt{4t^2 + 25 + 4t^2 - 64t + 256}$$

$$= \sqrt{8t^2 - 64t + 281}$$

The speed will be minimum / maximum when

$\frac{d}{dt}\|\vec{r}\,'(t)\| = 0$, so

$$\frac{d}{dt}\|\vec{r}\,'(t)\| = \frac{16t-64}{2\sqrt{8t^2-64t+281}} = \frac{8t-32}{\sqrt{8t^2-64t+281}}$$

which will be zero only when

$$8t - 32 = 0 \Rightarrow \underline{t = 4}$$

At $t=4$, the particle will be moving with minimum speed. The speed is not maximum at $t=4$ because if $t<4$, the speed is decreasing, while if $t>4$, the speed is increasing.

4. If $\vec{a}(t) = \langle 2t, \sin t, \cos 2t \rangle$, then

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \left\langle 2\frac{t^2}{2}, -\cos t, \frac{1}{2}\sin(2t) \right\rangle + \vec{C}$$

Since $\vec{v}(0) = \langle 1, 0, 0 \rangle$, we have

$$\vec{v}(0) = \langle 0, -1, 0 \rangle + \vec{C} = \langle 1, 0, 0 \rangle \Rightarrow \vec{C} = \langle 1, 1, 0 \rangle$$

So

$$\vec{v}(t) = \left\langle t^2 + 1, 1 - \cos t, \frac{1}{2}\sin(2t) \right\rangle$$

Now, the position vector $\vec{r}(t) = \int \vec{v}(t)\,dt$, so

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \frac{t^3}{3} + t, t - \sin t, -\frac{1}{4}\cos(2t) \right\rangle + \vec{D}$$

Since $\vec{r}(0) = \langle 0,1,0 \rangle$, we have

$$\vec{r}(0) = \left\langle 0, 0, -\frac{1}{4} \right\rangle + \vec{D} = \langle 0,1,0 \rangle$$

$$\Rightarrow \vec{D} = \left\langle 0, 1, \frac{1}{4} \right\rangle$$

$\therefore$

$$\underline{\vec{r}(t) = \left\langle \frac{t^3}{3} + t,\ t - \sin t + 1,\ \frac{1}{4} - \frac{1}{4}\cos(2t) \right\rangle}$$

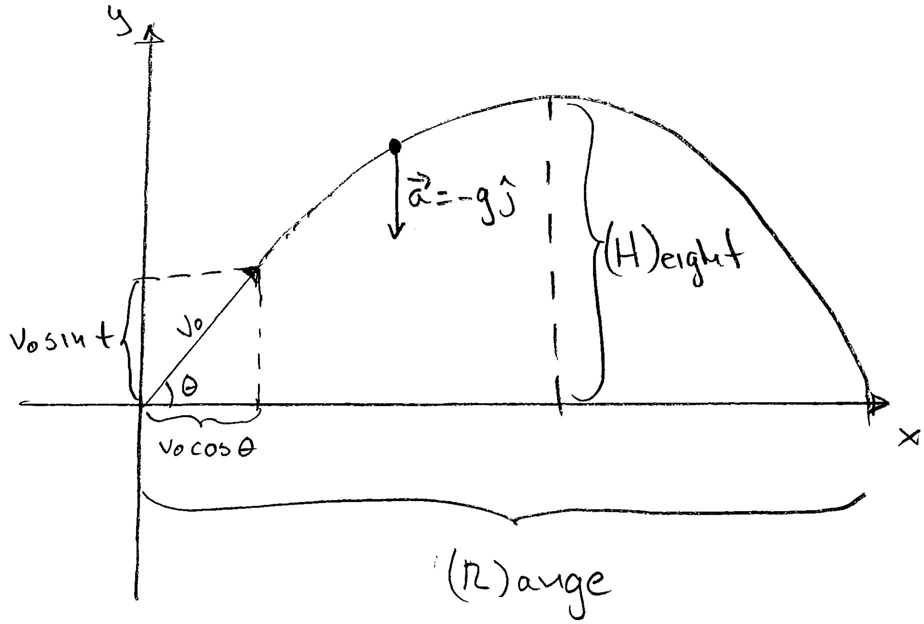
5. Since $\vec{F} = m\vec{a}$, we have that $\vec{F} = m\vec{r}''$, so

$$\vec{r}'(t) = \langle 3t^2, 2t, 3t^2 \rangle$$
$$\vec{r}''(t) = \langle 6t, 2, 6t \rangle$$

So the force is

$$\vec{F}(t) = m\vec{a}(t) = m\vec{r}''(t) = m\langle 6t, 2, 6t \rangle$$
$$= \underline{\langle 6mt, 2m, 6mt \rangle}$$

6. Consider the following figure, which we will use for tackling all 3 items:



a) If the initial speed of the launched object is v_0, then the initial velocity of the object is $\vec{v}(0) = \langle v_0 \cos\theta, v_0 \sin\theta \rangle$, where θ is the angle at which the object is launched.

Now, $\vec{a}(t) = -g\hat{j}$, where g is the standard acceleration constant.

Thus, $\vec{v}(t) = \int \vec{a}(t)\,dt = -gt\hat{j} + \vec{c}$. Since $\vec{v}(0) = \langle v_0 \cos\theta, v_0 \sin\theta \rangle$, we have

$$\vec{v}(t) = v_0 \cos\theta\,\hat{i} + (v_0 \sin\theta - gt)\,\hat{j} \qquad (1)$$

The position function is then $\vec{r}(t) = \int \vec{v}(t)\,dt$.

So

$$\vec{r}(t) = v_0 \cos\theta\, t\, \hat{i} + \left(v_0 \sin\theta\, t - \frac{g}{2} t^2\right)\hat{j} + \vec{D}$$

Since $\vec{r}(0) = \langle 0, 0\rangle$, $\vec{D} = \vec{0}$. Therefore,

$$\vec{r}(t) = v_0 \cos\theta\, t\, \hat{i} + \left(v_0 \sin\theta\, t - \frac{g}{2} t^2\right)\hat{j} \quad (2)$$

An object thrown from the ground will reach its maximum height when the component of its velocity in the positive y direction is equal to zero. This condition can be written as $\vec{v}(t)\cdot\hat{j} = 0$

So we have:

$$v_0 \sin\theta - g t_h = 0, \quad \text{or} \quad t_h = \frac{v_0 \sin\theta}{g} \quad (3)$$

If we substitute th in (2) we will be able to determine the height of the object as a function of the angle θ:

$$\vec{r}(t_h) = v_0 \cos\theta \left(\frac{v_0 \sin\theta}{g}\right)\hat{i} + v_0 \sin\theta \left(\frac{v_0 \sin\theta}{g}\right) - \frac{g}{2}\left(\frac{v_0 \sin\theta}{g}\right)^2 \hat{j}$$

Then, the height of the object is

$$h(\theta) = \frac{V_0^2 \sin^2\theta}{g} - \frac{g}{2}\frac{V_0^2 \sin^2\theta}{g^2}$$

$$= \frac{V_0^2 \sin^2\theta}{2g}$$

The maximum / minimum height is reached when

$\frac{dh}{d\theta} = 0$; therefore

$$\frac{dh}{d\theta} = \frac{V_0^2}{2g}(2\sin\theta\cos\theta) = \frac{V_0^2}{g}\sin\theta\cos\theta = 0 \quad (4)$$

Eq. 4 is solved when $\theta = \frac{\pi}{2}$ or $\theta = 0$. We see then that the maximum height is reached when $\theta = \frac{\pi}{2}$ (or when the object is vertically shot).

b) The maximum range will be reached when the component of $\vec{r}(t)$ in the positive z-direction is zero. Therefore:

$$V_0 \sin\theta \, t_r - \frac{g}{2} t_r^2 = 0$$

where t_r is the time when the object is on the ground.

$$t_r \left(V_0 \sin\theta - \frac{g}{2} t_r \right) = 0$$

$$\Rightarrow t_r = 0 \quad \text{or} \quad t_r = \frac{2 V_0 \sin\theta}{g} \qquad (5)$$

(5) in $\hat{i}$ component in (2)

$$V_0 \cos\theta \left(\frac{2 V_0 \sin\theta}{g} \right) = \text{range}$$

$$\frac{2 V_0^2 \sin\theta \cos\theta}{g} = \text{range}$$

The range will be maximum when $\sin\theta\cos\theta = \frac{1}{2}$ which occurs when $\theta = \frac{\pi}{4}$

c) To find the angle that maximizes the length of the object's trajectory, we use the arc length formula:

$$s = \int_a^b \|\vec{r}\,'(t)\|\, dt$$

$$\vec{r}\,'(t) = \vec{v}(t) = \langle v_0 \cos\theta,\ v_0 \sin\theta - gt \rangle$$

$$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{v_0^2 \cos^2\theta + (v_0 \sin\theta - gt)^2}$$

$$= \sqrt{v_0^2 \cos^2\theta + v_0^2 \sin^2\theta - 2v_0 g t \sin\theta + g^2 t^2}$$

$$= \sqrt{v_0^2 - 2v_0 g t \sin\theta + g^2 t^2}$$

Factorizing g^2:

$$\|\vec{r}\,'(t)\| = g\sqrt{\frac{v_0^2}{g^2} - \frac{2v_0 \sin\theta\, t}{g} + t^2}$$

Completing squares:

$$\|\vec{r}\,'(t)\| = g\sqrt{\frac{v_0^2}{g^2} - \frac{v_0^2 \sin^2\theta}{g^2} + \left(t - \frac{v_0 \sin\theta}{g}\right)^2}$$

$$\|\vec{r}\,'(t)\| = g\sqrt{\frac{v_0^2\cos^2\theta}{g^2} + \left(t - \frac{v_0\sin\theta}{g}\right)^2}$$

So,

$$S = \int_a^b g\sqrt{\frac{v_0^2\cos^2\theta}{g^2} + \left(t - \frac{v_0\sin\theta}{g}\right)^2}\,dt$$

where $a = 0$ (object on the ground @ $t = 0$)

$b = \frac{2v_0\sin\theta}{g}$ (object on the ground @ $t = t_r$, Eq. 5)

Thus:

$$S = \int_0^{\frac{2v_0\sin\theta}{g}} g\sqrt{\frac{v_0^2\cos^2\theta}{g^2} + \left(t - \frac{v_0\sin\theta}{g}\right)^2}\,dt$$

Using:

$$\int\sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right) + C$$

$$S = g\left[\frac{\left(t-\frac{v_0\sin\theta}{2}\right)}{2}\sqrt{\frac{v_0^2\cos^2\theta}{g^2}+\left(t-\frac{v_0\sin\theta}{g}\right)^2} + \frac{v_0^2\cos^2\theta}{2g^2}\ln\left(t-\frac{v_0\sin\theta}{g}+\sqrt{\frac{v_0^2\cos^2\theta}{g^2}+\left(t-\frac{v_0\sin\theta}{g}\right)^2}\right)\right]_0^{\frac{2v_0\sin\theta}{g}}$$

Simplifying:

$$S = \frac{g}{2}\left[\frac{v_0^2\sin\theta}{g^2}+\frac{v_0^2\cos^2\theta}{g^2}\ln\left(\frac{v_0(\sin\theta+1)}{g}\right)\right] - \frac{g}{2}\left[-\frac{v_0^2\sin\theta}{g^2}+\frac{v_0^2\cos^2\theta}{g^2}\ln\left(\frac{v_0(1-\sin\theta)}{g}\right)\right]$$

$$S = \frac{v_0^2\sin\theta}{g}+\frac{v_0^2\cos^2\theta}{2g}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$$

S will be maximum/minimum when $\frac{dS}{d\theta}=0$

So,

$$\frac{dS}{d\theta}=\frac{v_0^2\cos\theta}{g}\left(2-\sin\theta\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right)=0$$

If we restrict $0 < \theta < \frac{\pi}{2}$, then $\cos\theta \neq 0$ and $\frac{dS}{d\theta} = 0$ only when

$$2 - \sin\theta \ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = 0$$

Solving this equation numerically, we have

$$\theta = 0.9855 \text{ rad} \quad \text{or} \quad \theta = 56.43^\circ$$

7. $S = \int_0^1 \|\vec{r}\,'(t)\|\,dt$, where $\vec{r}(t) = \langle 2t, t^2, t^3/3 \rangle$ and $\vec{r}\,'(t) = \langle 2, 2t, t^2 \rangle$

so $\|\vec{r}\,'(t)\| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2+2)^2} = t^2 + 2$

and

$$S = \int_0^1 (t^2+2)\,dt = \left[\frac{t^3}{3} + 2t\right]_0^1 = \frac{1}{3} + 2 = \frac{7}{3}$$

8. $\vec{r}(t) = \langle \cos t, \sin t, \ln(\cos t) \rangle$

$$\vec{r}\,'(t) = \left\langle -\sin t, \cos t, \frac{-\sin t}{\cos t} \right\rangle$$

$$= \langle -\sin t, \cos t, -\tan t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$$

$$= \sqrt{1 + \tan^2 t}$$

$$= \sqrt{\sec^2 t} = |\sec t|, \text{ but since } 0 \le t \le \pi/4$$

$$= \sec t$$

So,

$$S = \int_0^{\pi/4} \sec t \, dt = \ln|\sec t + \tan t| \Big]_0^{\pi/4}$$

$$= \ln|\sqrt{2} + 1| - \cancel{\ln|1 + 0|}^{\,0}$$

$$\underline{= \ln(\sqrt{2} + 1) = 0.881}$$

9. $$s(t) = \int_a^t \|\vec{r}\,'(u)\|\,du$$

$$\vec{r}(u) = \langle 2u, 1-3u, 5+4u \rangle$$

$$\vec{r}\,'(u) = \langle 2, -3, 4 \rangle$$

$$\|\vec{r}\,'(u)\| = \sqrt{4+9+16} = \sqrt{29}$$

$$s(t) = \int_0^t \sqrt{29}\,du = \sqrt{29}\,u \Big]_0^t = \sqrt{29}\,t$$

$$\therefore \quad t = \frac{s}{\sqrt{29}}$$

The reparametrization is thus

$$\vec{r}(s) = \left\langle \frac{2s}{\sqrt{29}},\ 1 - \frac{3s}{\sqrt{29}},\ 5 + \frac{4s}{\sqrt{29}} \right\rangle$$

10. $\vec{r}(u) = \langle 2\cos u,\ 2\sin u,\ u \rangle$

$\vec{r}\,'(u) = \langle -2\sin u,\ 2\cos u,\ 1 \rangle$

$\|\vec{r}\,'(u)\| = \sqrt{4\sin^2 u + 4\cos^2 u + 1}$

$= \sqrt{4+1} = \sqrt{5}$

$s(t) = \int_0^t \sqrt{5}\, du = \sqrt{5}\, t \Rightarrow t = \frac{s}{\sqrt{5}}$

So,

$\underline{\vec{r}(s) = \left\langle 2\cos\left(\frac{s}{\sqrt{5}}\right),\ 2\sin\left(\frac{s}{\sqrt{5}}\right),\ \frac{s}{\sqrt{5}} \right\rangle}$

$\underline{\vec{r}(\sqrt{5}) = \langle 2\cos(1),\ 2\sin(1),\ 1 \rangle}$

Thus moving $\sqrt{5}$ units along the curve would take us to $(1.08, 1.68, 1)$

$\underline{\vec{r}(4) = \left\langle 2\cos\left(\frac{4}{\sqrt{5}}\right),\ 2\sin\left(\frac{4}{\sqrt{5}}\right),\ \frac{4}{\sqrt{5}} \right\rangle}$

Moving 4 units along the curve would take us to $(-0.43, 1.95, 1.78)$