Homework #6

Math 243 - Section 51

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1. $\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$, since $\vec{r}(s)$ is already expressed in terms of arc length, we just need to find $\vec{T}(s)$ and differentiate with respect to s.

$$\vec{T}(s) = \frac{\vec{r}'(s)}{\|\vec{r}'(s)\|} = \frac{1}{2\sqrt{2}} \langle 2, -2 \rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$$

$$\frac{d\vec{T}(s)}{ds} = \vec{0} \Rightarrow \kappa = \|\vec{0}\| = 0.$$

It is easy to verify that this result is correct because $\vec{r}(s)$ represents a straight line.

2. Let us use the expression $\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$

So, $\vec{r}'(t) = \langle -2\pi \sin \pi t, \pi \cos \pi t \rangle$

$$\|\vec{r}\,'(t)\| = \sqrt{(-2\pi \sin(\pi t))^2 + (\pi \cos \pi t)^2}$$

$$= \sqrt{4\pi^2 \sin^2 \pi t + \pi^2 \cos^2 \pi t}$$

$$= \pi \sqrt{3\sin^2 \pi t + \underbrace{\sin^2 \pi t + \cos^2 \pi t}_{1}}$$

$$= \pi \sqrt{3 \sin^2 \pi t + 1}$$

$$\vec{r}\,''(t) = \langle -2\pi^2 \cos \pi t, -\pi^2 \sin \pi t \rangle$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\pi \sin \pi t & \pi \cos \pi t & 0 \\ -2\pi^2 \cos \pi t & -\pi^2 \sin \pi t & 0 \end{vmatrix}$$

$$= 0\hat{i} - 0\hat{j} + (2\pi^3 \sin^2 \pi t + 2\pi^3 \cos^2 \pi t)\hat{k}$$

$$= 2\pi^3 \hat{k}$$

$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = 2\pi^3$$

$$\kappa(t) = \frac{2\pi^3}{\left(\pi\sqrt{3\sin^2 \pi t + 1}\right)^3} = \boxed{\frac{2}{(3\sin^2 \pi t + 1)^{3/2}}}$$

3. Using $\kappa(t) = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$, we have

$\vec{r}\,'(t) = \langle 1, 2t, t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1 + 4t^2 + t^2} = \sqrt{1 + 5t^2}$

$\vec{r}\,''(t) = \langle 0, 2, 1 \rangle$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & t \\ 0 & 2 & 1 \end{vmatrix} = (2t - 2t)\hat{i} - (1 - 0)\hat{j} + (2 - 0)\hat{k} = 0\hat{i} - \hat{j} + 2\hat{k}$$
$$= \langle 0, -1, 2 \rangle$$

$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \sqrt{1 + 4} = \sqrt{5}$

So $\kappa(t) = \dfrac{\sqrt{5}}{(1 + 5t^2)^{3/2}}$

4. A parabola that passes through the origin has equation $y = Ax^2$ or $x = Ay^2$. Let us use $y = Ax^2$.

The vector equation that represents this parabola is

$$\vec{r}(t) = \langle t, At^2, 0 \rangle$$

So, if $\kappa(t) = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$, then

$$\vec{r}\,'(t) = \langle 1, 2At, 0 \rangle$$

$$\vec{r}\,''(t) = \langle 0, 2A, 0 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + 4A^2t^2}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2At & 0 \\ 0 & 2A & 0 \end{vmatrix} = 2A\hat{k}$$

$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = |2A|$$

$\kappa(t) = \dfrac{|2A|}{(1 + 4A^2t^2)^{3/2}}$; however, at the origin,

$t = 0$, so $\kappa(0) = |2A| = \dfrac{1}{2}$ (by the problem's statement.)

$\Rightarrow (|2A|)^2 = \left(\dfrac{1}{2}\right)^2 \Rightarrow 4A^2 = \dfrac{1}{4} \Rightarrow A^2 = 1 \Rightarrow$
$|A| = 1$ or $\boxed{A = \pm 1}$

Therefore, the equation of a parabola with curvature $\frac{1}{2}$ at the origin is

$$y = x^2 \quad \text{or} \quad y = -x^2$$

5. $\vec{r}(t) = \langle \sinh t, \cosh t, t \rangle$

$\vec{r}\,'(t) = \langle \cosh t, \sinh t, 1 \rangle$

$\vec{r}\,''(t) = \langle \sinh t, \cosh t, 0 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{\cosh^2 t + \sinh^2 t + 1}$

But $\cosh^2 x + \sinh^2 x =$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 =$$

$$\frac{(e^x + e^{-x})^2 + (e^x - e^{-x})^2}{4} =$$

$$\frac{e^{2x} + 2e^x e^{-x} + e^{-2x} + e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} =$$

$$\frac{2e^{2x} + 2e^{-2x}}{4} = \frac{1}{2}\left(e^{2x} + e^{-2x}\right) = \cosh 2x$$

So

$$\|\vec{r}\,'(t)\| = \sqrt{\cosh 2t + 1}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cosh t & \sinh t & 1 \\ \sinh t & \cosh t & 0 \end{vmatrix}$$

$$= (-\cosh t)\hat{i} - (-\sinh t)\hat{j} + (\cosh^2 t - \sinh^2 t)\hat{k}$$

$$= \langle -\cosh t, \sinh t, \cosh^2 t - \sinh^2 t \rangle$$

But

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 =$$

$$\frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2e^x e^{-x} + e^{-2x} - (e^{2x} - 2e^x e^{-x} + e^{-2x})}{4}$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4} = \frac{4}{4} = 1$$

Therefore

$$\vec{r}' \times \vec{r}'' = \langle -\cosh t, \sinh t, 1 \rangle$$

and

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{\cosh^2 t + \sinh^2 t + 1} = \|\vec{r}'(t)\|$$

$$= \sqrt{\cosh 2t + 1}$$

$$\kappa(t) = \frac{\sqrt{\cosh 2t + 1}}{(\cosh 2t + 1)^{3/2}} \quad \boxed{\frac{1}{\cosh 2t + 1}}$$

At $(0,1,0)$, $t=0$, so

$$\kappa(0) = \frac{1}{\cosh 0 + 1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

6. A function on a plane, say $y=f(x)$, can be parametrized as

$$\vec{r}(t) = \langle \underset{"x"}{t}, \underset{"y"}{f(t)}, \underset{"z"}{0} \rangle$$

So its curvature is given by $\kappa(t) = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$

$\vec{r}\,'(t) = \langle 1, f'(t), 0 \rangle$; $\vec{r}\,''(t) = \langle 0, f''(t), 0 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1 + (f'(t))^2}$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix} = f''(t)\,\hat{k}$$

$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = |f''(t)|$

Therefore

$$\kappa(t) = \frac{|f''(t)|}{\left(1 + (f'(t))^2\right)^{3/2}}$$

At an inflection point, $f''(t) = 0$, therefore $\kappa(t) = 0$.

7. The centripetal force exerted on the car is proportional to the normal component of the car's acceleration. That is

$$a_{centripetal} = K(t)\left(\frac{ds}{dt}\right)^2.$$

So, if the nominal speed is $\frac{ds}{dt}$, then doubling it produces

$$a_{centripetal} = K(t)\left(2\frac{ds}{dt}\right)^2 = 4\,K(t)\left(\frac{ds}{dt}\right)^2$$
$$= 4\,a_{nominal}.$$

Therefore, the force is increased by a factor of 4.

8. For a satellite to stay in orbit, it has to cancel the gravitational force of the Earth. Thus, the normal component of its acceleration must be equal to the gravitational acceleration due to the Earth.

Now, the magnitude of the gravitational force of the Earth is given by

$$\frac{GMm}{r^2} = \|\vec{F}_{gravitational}\|$$

where G is the gravitational constant ($6.67 \times 10^{-11}\ m^3/kg\,s^2$), M is the mass of the Earth ($\approx 5.972 \times 10^{24}\ kg$), m is the mass of the satellite (unknown), and r is the distance between the satellite and the center of the Earth (avg. Earth radius $\approx 6371 \times 10^3\ m$ + satellite's orbit distance $= 600$ miles $\approx 965 \times 10^3\ m$, for a total of $7.336 \times 10^6\ m$).

This force must be equal to the centrifugal force whose magnitude is given by

$$\|\vec{F}_{centrifugal}\| = mK\left(\frac{ds}{dt}\right)^2$$

where m is the mass of the satellite, κ is the curvature of the satellite's trajectory which we know is a circle of radius r, and so $\kappa = \frac{1}{r}$, and finally $\frac{ds}{dt}$ is the satellite's speed which we wish to find.

So

$$\frac{GM\cancel{m}}{r^2} = \cancel{m}\kappa\left(\frac{ds}{dt}\right)^2$$

$$\frac{GM}{r^2} = \frac{1}{r}\left(\frac{ds}{dt}\right)^2$$

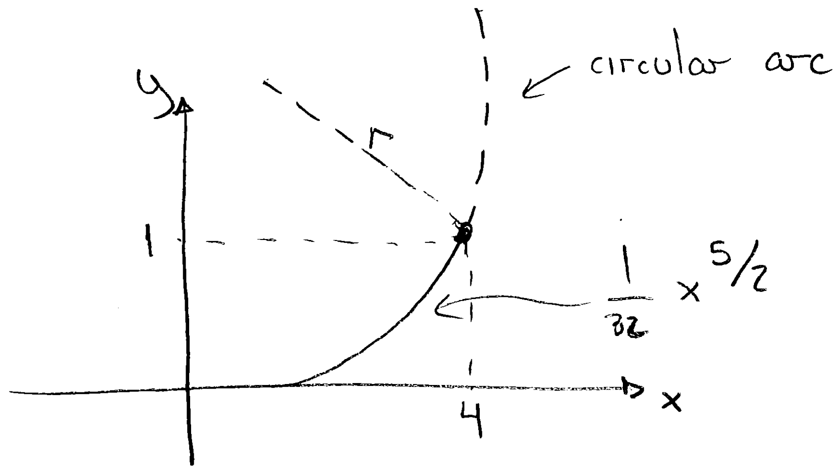
$$\frac{GM}{r} = \left(\frac{ds}{dt}\right)^2$$

$$\sqrt{\frac{GM}{r}} = \frac{ds}{dt}$$

Substituting values for G, M, and r, we obtain

$$\frac{ds}{dt} \approx 7368 \text{ m/s} \approx 4.57 \text{ miles/s}$$

9.



The radius of the circular arc, r, is the reciprocal of the circle's curvature.* So $r = \frac{1}{\kappa}$ or $\kappa = \frac{1}{r}$.

Thus, by knowing the curvature at (4, 1), we can find the arc's radius.

We are told that κ at (4, 1) is equal to the curvature of $y = \frac{1}{32}x^{5/2}$ at (4, 1).

Using the result of exercise 6, we know that

$$\kappa = \frac{|f''(x)|}{\left(1+(f'(x))^2\right)^{3/2}}$$

So, $f'(x) = \frac{1}{32}\left(\frac{5}{2}\right)x^{3/2} = \frac{5}{64}x^{3/2}$

* We proved this in class.

$$f''(x) = \frac{5}{64}\left(\frac{3}{2}\right) x^{1/2} = \frac{15}{128}\, x^{1/2}$$

So

$$K = \frac{\left|\frac{15}{128}\, x^{1/2}\right|}{\left[1 + \left(\frac{5}{64}\, x^{3/2}\right)^2\right]^{3/2}} = \frac{\frac{15}{128}\, x^{1/2}}{\left[1 + \frac{25}{4096}\, x^3\right]^{3/2}}$$

at (4,1):

$$K = \frac{\frac{15}{128}(2)}{\left[1 + \frac{25}{4096}(64)\right]^{3/2}} = \frac{\frac{30}{128}}{\left[1 + \frac{25(2)}{128}\right]^{3/2}} = \frac{\frac{15}{64}}{\left[\frac{128+50}{128}\right]^{3/2}}$$

$$= \frac{\frac{15}{64}}{\left(\frac{178}{128}\right)^{3/2}} \Rightarrow r = \frac{1}{K} = \frac{\left(\frac{178}{128}\right)^{3/2}}{\frac{15}{64}} \approx 7$$

Both curves should have the same curvature so that the acceleration (in the normal direction) doesn't change abruptly when passing through the connection.

10. $\vec{r}'(t) = \|\vec{r}'(t)\|\,\vec{T}(t) = \dfrac{ds}{dt}\vec{T}(t)$

$$\vec{r}''(t) = \frac{d^2s}{dt^2}\vec{T}(t) + \kappa(t)\left(\frac{ds}{dt}\right)^2 N(t)$$

$$\vec{r}'(t)\times\vec{r}''(t) = \frac{ds}{dt}\vec{T}(t) \times \left(\frac{d^2s}{dt^2}\vec{T}(t) + \kappa(t)\left(\frac{ds}{dt}\right)^2\vec{N}(t)\right)$$

$$= \left(\frac{ds}{dt}\vec{T}(t)\times\frac{d^2s}{dt^2}\vec{T}(t)\right) + \left(\frac{ds}{dt}\vec{T}(t)\times\kappa(t)\left(\frac{ds}{dt}\right)^2\vec{N}(t)\right)$$

$$= \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\vec{T}(t)\times\vec{T}(t) + \kappa(t)\left(\frac{ds}{dt}\right)^3\vec{T}(t)\times\vec{N}(t)$$

Since $\vec{T}(t)$ is parallel to itself; $\vec{T}\times\vec{T} = \vec{0}$

$$\vec{r}'(t)\times\vec{r}''(t) = \kappa(t)\left(\frac{ds}{dt}\right)^3\vec{T}(t)\times\vec{N}(t) = \kappa(t)\left(\|\vec{r}'(t)\|\right)^3\vec{T}(t)\times\vec{N}(t)$$

Obtaining its norm:

$$\|\vec{r}'(t)\times\vec{r}''(t)\| = \|\kappa(t)\left(\|\vec{r}'(t)\|\right)^3\vec{T}(t)\times\vec{N}(t)\|$$

$$= \kappa(t)\,\|\vec{r}'(t)\|^3\,\underbrace{\|\vec{T}(t)\times\vec{N}(t)\|}_{\cancel{\|\vec{T}(t)\|}^{1}\,\cancel{\|\vec{N}(t)\|}^{1}\,\cancel{\sin\theta}^{1}}$$

$$\theta = 90^\circ$$

$$= \kappa(t)\,\|\vec{r}'(t)\|^3$$

$$\Rightarrow \boxed{\kappa(t) = \frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}}$$