Homework #7
Math 243 - Section 51
Dr. Marco A. Montes de Oca

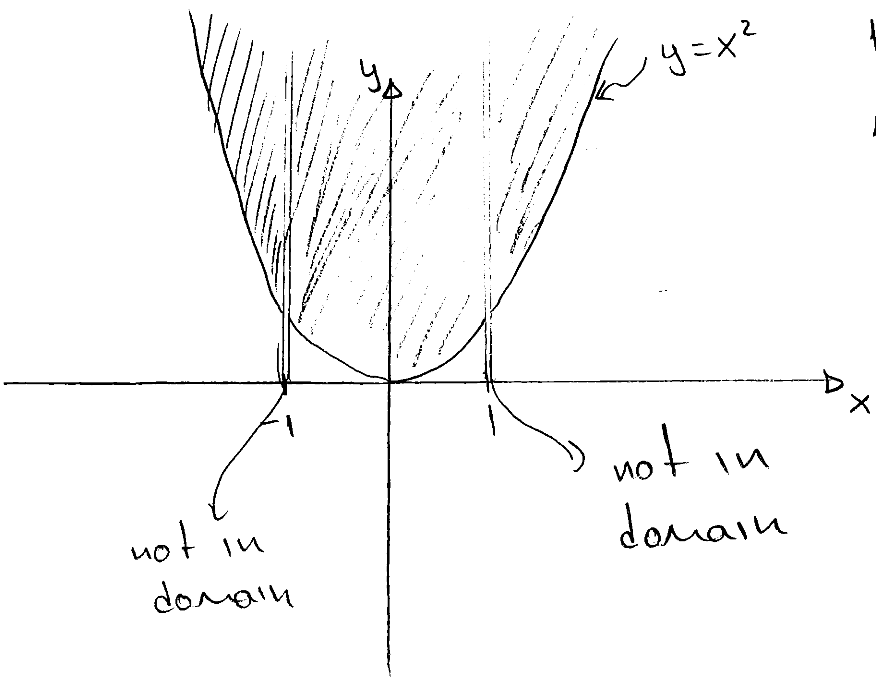
1. $f(x,y) = \dfrac{\sqrt{y-x^2}}{1-x^2}$ is a well-defined real number when $y-x^2 \geq 0$ and $1-x^2 \neq 0$.

This means that $y \geq x^2$ and $1 \neq x^2 \Rightarrow$

$1 \neq \sqrt{x^2} \Rightarrow$

$1 \neq |x| \Rightarrow$

$x \neq 1$ or $x \neq -1$



2. $f(x,y) = \int_x^y (2t + \sin(t))\, dt$

$$= \left[t^2 - \cos(t)\right]_x^y = y^2 - \cos y - (x^2 - \cos x)$$

$$= y^2 - x^2 - \cos y + \cos x$$

So $f(0,1) = 1^2 - \cos(1) = \underline{1 - \cos(1)}$

$f(4,\pi) = \pi^2 - 16 - \cos(\pi) + \cos(4)$

$$= \pi^2 - 16 + (-1) + \cos(4) = \underline{\pi^2 - 17 + \cos(4)}$$

3. See attached plots.

The graph of $f(x,y) = g(\sqrt{x^2+y^2})$ is obtained from the graph of g by revolving it around the z-axis.

4. $\lim_{(x,y)\to(0,0)} \dfrac{x+y}{x^2+y}$ DNE

because if we approach $(0,0)$ through the y-axis we obtain $f(0,y) = 1$, therefore

$f(x,y) \to 1$ as $(x,y) \to (0,0)$ along the y-axis additionally, if we approach $(0,0)$ along the x-axis we obtain $f(x,0) = \frac{1}{x}$, therefore

$f(x,y)$ DNE as $(x,y) \to (0,0)$ along the x-axis

5. $\lim\limits_{(x,y)\to(1,1)} \frac{xy-1}{xy+1}$

Lines of different slope that pass through $(1,1)$ are given by $y = m(x-1)+1$, so if we take one of those lines to approach $(1,1)$, we can transform

$$\frac{xy-1}{xy+1} = \frac{x(mx-m+1)-1}{x(mx-m+1)+1} = \frac{mx^2-mx+x-1}{mx^2-mx+x+1}$$

as $x \to 1$, $\frac{mx^2-mx+x-1}{mx^2-mx+x+1} \to \frac{\cancel{m-m}+1-1}{\cancel{m-m}+1+1} \to 0$
(independently of m)

$\therefore \lim\limits_{(x,y)\to(1,1)} \frac{xy-1}{xy+1} = 0$

6. $$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{\underbrace{x^2+y^2+z^2}_{f(x,y,z)}}$$

If we approach $(0,0,0)$ along the x-axis, then

$$f(x,0,0) = \frac{0}{x^2} = 0 \Rightarrow \lim_{(x,0,0)\to(0,0,0)} f(x,0,0) = 0.$$

Now, if we approach $(0,0,0)$ on the xy-plane along $y=x$.

$$f(x,x,0) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

$$\therefore \lim_{(x,x,0)\to(0,0,0)} f(x,x,0) = \frac{1}{2} \neq 0$$

$$\therefore \lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} \quad \text{DNE.}$$

7. $$f(x,y) = \frac{e^y}{x+y^2}$$

$$\frac{\partial}{\partial x} f(x,y) = e^y \frac{\partial}{\partial x}\left(\frac{1}{x+y^2}\right) = e^y \frac{\partial}{\partial x}\left((x+y^2)^{-1}\right)$$

$$e^y(-1)(x+y^2)^{-2}(1) = -e^y(x+y^2)^{-2} = \underline{\frac{-e^y}{(x+y^2)^2}}$$

$$\frac{\partial}{\partial y} f(x,y) = \frac{(x+y^2)e^y - e^y(2y)}{(x+y^2)^2} = \underline{\frac{e^y(x+y^2-2y)}{(x+y^2)^2}}$$

If $f(x,y) = \int_x^y \cos(e^t)\,dt$

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} \int_x^y \cos(e^t)\,dt = \frac{\partial}{\partial x}\left[-\int_y^x \cos(e^t)\,dt\right]$$

[Using the FTC]

$= \underline{-\cos(e^x)}$

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} \int_x^y \cos(e^t)\,dt = \underline{\cos(e^y)}$$

8. $f(x,y) = x^3y^5 + 2x^4y$

$$\frac{\partial f}{\partial x} = 3x^2y^5 + 8x^3y \quad ; \quad \frac{\partial f}{\partial y} = 5x^3y^4 + 2x^4$$

Keep scrolling down.

$$\frac{\partial^2 f}{\partial x^2} = \underline{6xy^5 + 24x^2y}$$

$$\frac{\partial^2 f}{\partial y^2} = \underline{20x^3y^3}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \underline{15x^2y^4 + 8x^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \underline{15x^2y^4 + 8x^3}$$

$$f(x,y) = \frac{xy}{x-y}$$

$$\frac{\partial f}{\partial x} = \frac{(x-y)y - xy}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x-y)x - xy(-1)}{(x-y)^2} = \frac{x^2}{(x-y)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = -y^2 \frac{\partial}{\partial x}\left((x-y)^{-2}\right) = -y^2(-2)(x-y)^{-3} = \underline{\frac{2y^2}{(x-y)^3}}$$

$$\frac{\partial^2 f}{\partial y^2} = x^2 \frac{\partial}{\partial y}\left((x-y)^{-2}\right) = x^2(-2)(x-y)^{-3}(-1)$$

$$= \frac{2x^2}{(x-y)^3}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-y^2}{(x-y)^2}\right) = \frac{(x-y)^2(-2y) - (-y^2)\left(2(x-y)(-1)\right)}{(x-y)^4}$$

$$= \frac{-2y(x-y)^2 - 2y^2(x-y)}{(x-y)^4}$$

$$= \frac{-2y\cancel{(x-y)}\left[(x\cancel{-y}) \cancel{+y}\right]}{\cancel{(x-y)^4}}$$

$$= \boxed{\frac{-2xy}{(x-y)^3}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x^2}{(x-y)^2}\right) = \frac{(x-y)^2(2x) - x^2\left(2(x-y)\right)}{(x-y)^4}$$

$$= \frac{2x\cancel{(x-y)}\left[(\cancel{x}-y) \cancel{-x}\right]}{\cancel{(x-y)^4}}$$

$$= \boxed{\frac{-2xy}{(x-y)^3}}$$

9. $F(x,y) = x^4y^2 - x^3y$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial x}\left(x^4y^2 - x^3y\right)\right)$$

$$= \frac{\partial^2}{\partial x^2}\left(4x^3y^2 - 3x^2y\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(4x^3y^2 - 3x^2y\right)\right) = \frac{\partial}{\partial x}\left(12x^2y^2 - 6xy\right)$$

$$= \underline{24xy^2 - 6y}$$

$$\frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial}{\partial x}\left(x^4y^2 - x^3y\right)\right)$$

$$= \frac{\partial^2}{\partial x \partial y}\left(4x^3y^2 - 3x^2y\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(4x^3y^2 - 3x^2y\right)\right)$$

$$= \frac{\partial}{\partial x}\left(8x^3y - 3x^2\right) = \underline{24x^2y - 6x}$$

10. $F(x,y) = e^{xy}\sin y$

$f_x = ye^{xy}\sin y$

$f_y = xe^{xy}\sin y + e^{xy}\cos y$

$$f_{xy} = \frac{\partial}{\partial y}\left(ye^{xy}\sin y\right)$$

$$= \frac{\partial}{\partial y}\left(ye^{xy}\right)\sin y + ye^{xy}\cos y$$

$$= \left(e^{xy} + xye^{xy}\right)\sin y + ye^{xy}\cos y$$

$$= \underline{e^{xy}\left(\sin y + xy\sin y + y\cos y\right)}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(xe^{xy}\sin y + e^{xy}\cos y\right)$$

$$= \sin y\left(e^{xy} + yxe^{xy}\right) + ye^{xy}\cos y$$

$$= \underline{e^{xy}\left(\sin y + xy\sin y + y\cos y\right)}$$

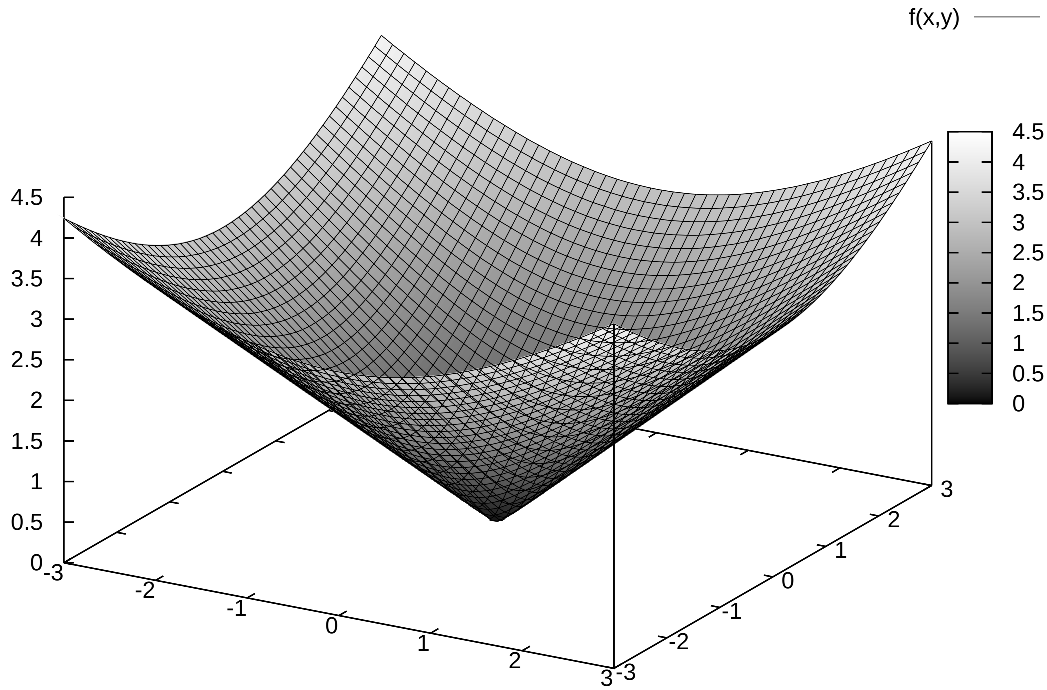


Figure 1: $z = \sqrt{x^2 + y^2}$

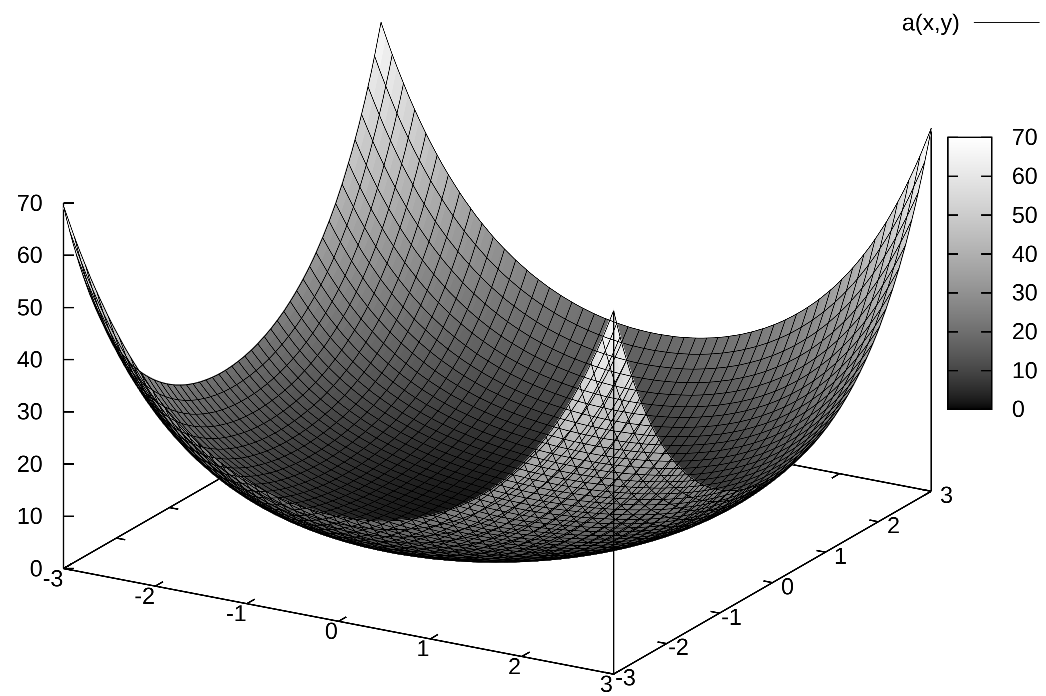


Figure 2: $z = e^{\sqrt{x^2+y^2}}$

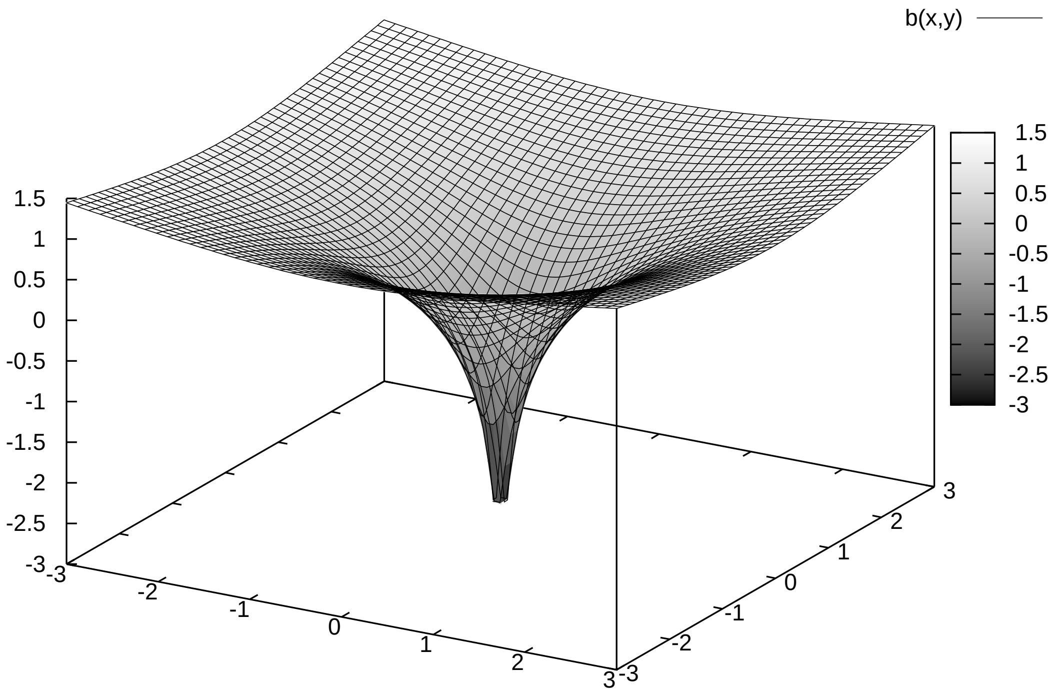


Figure 3: $z = \ln(\sqrt{x^2 + y^2})$

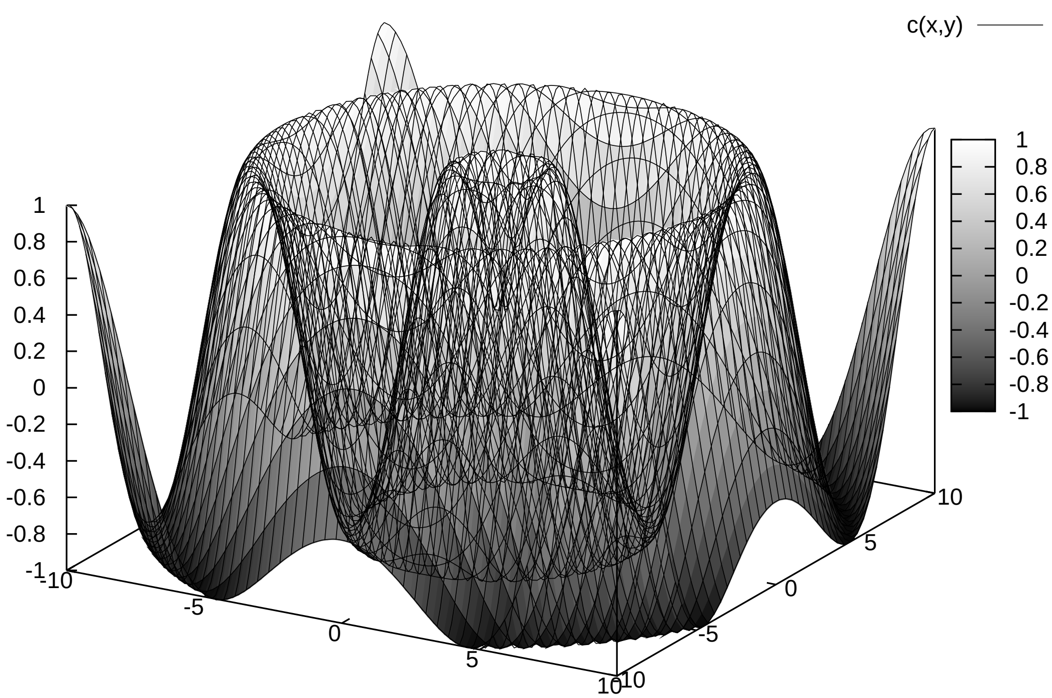


Figure 4: $z = \sin(\sqrt{x^2 + y^2})$

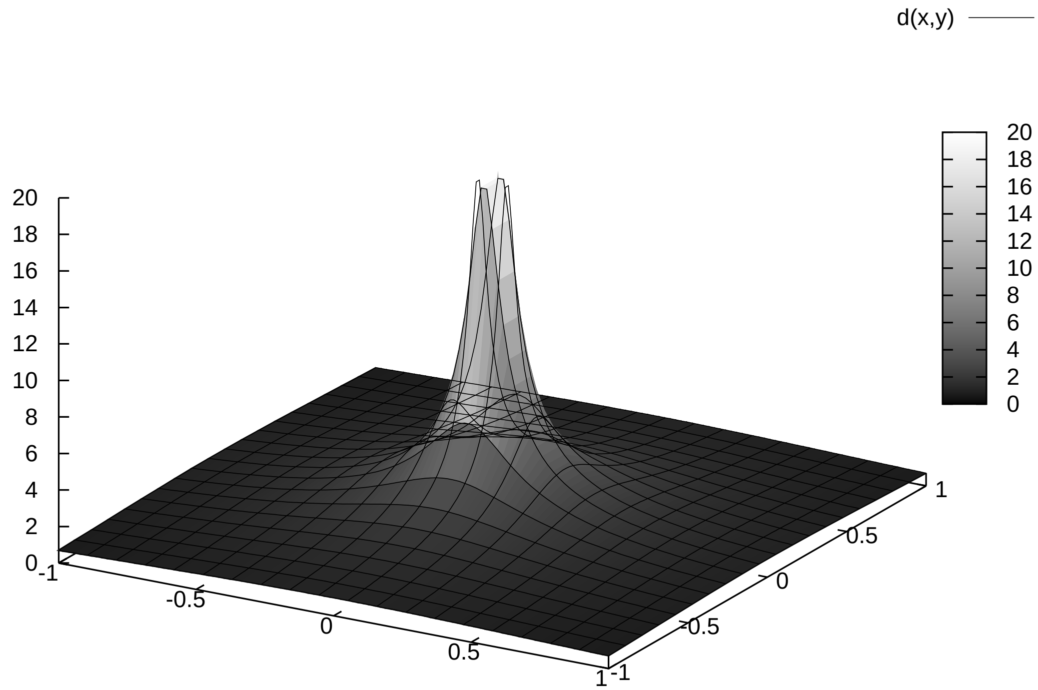


Figure 5: $z = \frac{1}{\sqrt{x^2+y^2}}$