Homework #9
Math 243 - Section 51
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1. $f(x,y) = e^x \cos y$

$f_x(x,y) = e^x \cos y$; $f_x(0,0) = e^0 \cos 0 = 1$

$f_y(x,y) = -e^x \sin y$; $f_y(0,0) = -e^0 \sin 0 = 0$

$\therefore \quad \nabla f(0,0) = \langle 1, 0 \rangle$

$\hat{u} = \langle \cos \frac{\pi}{4}, \sin \frac{\pi}{4} \rangle \Rightarrow$

$D_{\hat{u}} f(0,0) = \langle 1, 0 \rangle \cdot \langle \cos \frac{\pi}{4}, \sin \frac{\pi}{4} \rangle = \underline{\cos \frac{\pi}{4}}$

2. $f(x,y) = 2\sqrt{x} - y^2$

$f_x(x,y) = 2\left(\frac{1}{2} x^{-\frac{1}{2}}\right) = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$; $f_x(1,5) = 1$

$f_y(x,y) = 2y$; $f_y(1,5) = 10$

$\therefore \quad \nabla f(1,5) = \langle 1, 10 \rangle$

The direction is given by the vector

$\vec{v} = \langle 4-1, 1-5 \rangle = \langle 3, -4 \rangle \Rightarrow$

$\hat{v} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$

So

$D_{\hat{v}} f(1,5) = \langle 1, 10 \rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{3}{5} - \frac{40}{5}$

$= \boxed{\frac{-37}{5}}$

3. $f(x,y) = \dfrac{y^2}{x}$

$\nabla f(x,y) = \left\langle -\frac{y^2}{x^2}, \frac{2y}{x} \right\rangle$

$\nabla f(1,2) = \left\langle -\frac{(2)^2}{1}, \frac{2(2)}{1} \right\rangle = \langle -4, 4 \rangle$

$D_{\hat{v}} f(1,2) = \langle -4, 4 \rangle \cdot \left\langle \frac{2}{3}, \frac{\sqrt{5}}{3} \right\rangle = \frac{-8}{3} + \frac{4\sqrt{5}}{3} = \underline{\frac{4\sqrt{5} - 8}{3}}$

4. $\nabla f(x,y,z) = \langle y^2 e^{xyz}(yz),\ 2ye^{xyz} + y^2 e^{xyz}(xz),\ y^2 e^{xyz}(xy)\rangle$

$= \langle y^3 z e^{xyz},\ e^{xyz}(2y + xy^2 z),\ xy^3 e^{xyz}\rangle$

$\nabla f(0,1,-1) = \langle (1)^3(-1)e^0,\ e^0(2(1)+0),\ 0\rangle$

$= \langle -1, 2, 0\rangle$

Since $\| \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle \| = 1$, then

$D_{\hat{u}} f(0,1,-1) = \langle -1,2,0\rangle \cdot \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle = \frac{-3+8}{13} = \frac{5}{13}$

5. The maximum rate of change of $f(x,y,z)$ occurs in the direction of $\nabla f(x,y,z)$ and its magnitude is equal to $\|\nabla f(x,y,z)\|$.

$\nabla f(x,y,z) = \langle \frac{1}{z}, \frac{1}{z}, -\frac{(x+y)}{z^2}\rangle$

$\nabla f(1,1,-1) = \langle -1, -1, -2\rangle$

$\|\nabla f(1,1,-1)\| = \sqrt{6}$

6. The direction of maximum change is $\nabla f(x,y)$

So,

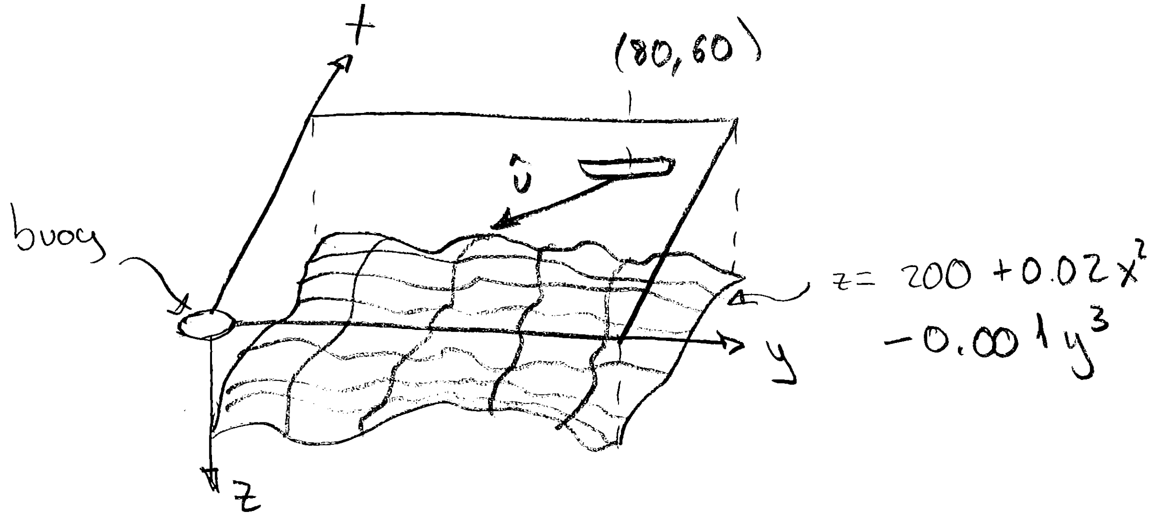
$$\nabla f(x,y) = \langle 2x-2,\ 2y-4 \rangle$$

The points (x,y) at which $\nabla f(x,y) = \langle 1,1 \rangle$ satisfy

$$\langle 2x-2,\ 2y-4 \rangle = \langle 1,\ 1 \rangle$$

$$\begin{cases} 2x-2=1 \\ 2y-4=1 \end{cases} \Rightarrow \begin{array}{l} 2x=3 \Rightarrow x=\frac{3}{2} \\ 2y=5 \Rightarrow y=\frac{5}{2} \end{array}$$

7.



$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}; \quad \vec{u} = \langle 0-80,\ 0-60 \rangle = \langle -80, -60 \rangle$$

$$\|\vec{u}\| = \sqrt{(-80)^2 + (-60)^2} = 100$$

$$\hat{u} = \langle -\frac{8}{10},\ -\frac{6}{10} \rangle$$

$$\nabla f(x,y) = \langle 2(0.02)\,x,\ -3(0.001)\,y^2 \rangle$$

$$= \langle 0.04\,x,\ -0.003\,y^2 \rangle$$

$$\nabla f(80,60) = \langle 3.2,\ -10.8 \rangle$$

$$\Rightarrow D_u f(80,60) = \langle 3.2,\ -10.8 \rangle \cdot \langle -\tfrac{8}{10},\ -\tfrac{6}{10} \rangle$$

$$= -2.56 + 6.48 = \underline{3.92}$$

$\therefore$ The depth is increasing, that is, the water under the boat is getting deeper.

8. The trajectory of the spaceship is described by the line

$$\vec{p}(t) = \langle x(t),\ y(t),\ z(t) \rangle$$

$$= \langle 2, 3, 1 \rangle + t\langle 3-2,\ 4-3,\ 3-1 \rangle$$

$$= \langle 2, 3, 1 \rangle + \langle t,\ t,\ 2t \rangle$$

$$= \langle 2+t,\ 3+t,\ 1+2t \rangle$$

However, we are told that the spaceship's speed $||\vec{r}'(t)|| = 5$, therefore

$$5 = ||\vec{r}'(t)|| = a\,||\vec{p}'(t)|| = a\,||\langle 1, 1, 2\rangle|| = a\sqrt{6}$$

$$\Rightarrow a = \frac{5}{\sqrt{6}}$$

So, the actual vector function that describes the movement of the spaceship is

$$\vec{r}(t) = \frac{5}{\sqrt{6}}\langle 2+t,\ 3+t,\ 1+2t\rangle$$

$$\Rightarrow x(t) = \frac{5}{\sqrt{6}}(2+t) \Rightarrow \frac{dx}{dt} = \frac{5}{\sqrt{6}}$$

$$y(t) = \frac{5}{\sqrt{6}}(3+t) \Rightarrow \frac{dy}{dt} = \frac{5}{\sqrt{6}}$$

$$z(t) = \frac{5}{\sqrt{6}}(1+2t) \Rightarrow \frac{dz}{dt} = \frac{10}{\sqrt{6}}$$

Now

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

$$= yz^3\left(\frac{5}{\sqrt{6}}\right) + xz^3\left(\frac{5}{\sqrt{6}}\right) + 3xyz^2\left(\frac{10}{\sqrt{6}}\right)$$

$$\left.\frac{dT}{dt}\right|_{(2,3,1)} = 3\left(\frac{5}{\sqrt{6}}\right) + 2\left(\frac{5}{\sqrt{6}}\right) + (2)(3)\left(\frac{30}{\sqrt{6}}\right) = \boxed{\frac{205}{\sqrt{6}}\ {}^{\circ}C/s}$$

9. Given $u(x,y)$ and $v(x,y)$, we can build a function $f(x,y) = a\,u(x,y) + b\,v(x,y)$ for any a, b, constants. So

$$\nabla f(x,y) = \langle a\,u_x(x,y) + b\,v_x(x,y),\ a\,u_y(x,y) + b\,v_y(x,y) \rangle$$

$$= \langle a\,u_x(x,y),\ a\,u_y(x,y) \rangle + \langle b\,v_x(x,y) + b\,v_y(x,y) \rangle$$

$$= a\langle u_x(x,y),\ u_y(x,y) \rangle + b\langle v_x(x,y),\ v_y(x,y) \rangle$$

$$= a\nabla u(x,y) + b\,\nabla v(x,y)$$

b) If $f(x,y) = u(x,y)\,v(x,y)$

$$\nabla f(x,y) = \langle u_x(x,y)\,v(x,y) + u(x,y)\,v_x(x,y),\ u_y(x,y)\,v(x,y) + u(x,y)\,v_y(x,y) \rangle$$

$$= \langle u_x(x,y)\,v(x,y),\ u_y(x,y)\,v(x,y) \rangle + \langle u(x,y)\,v_x(x,y),\ u(x,y)\,v_y(x,y) \rangle$$

$$= v(x,y)\,\nabla u(x,y) + u(x,y)\,\nabla v(x,y)$$

c) If $f(x,y) = \dfrac{u(x,y)}{v(x,y)}$, then

$$\nabla f(x,y) = \left\langle \frac{v(x,y)\,u_x(x,y) - u(x,y)\,v_x(x,y)}{v^2(x,y)}, \frac{v(x,y)\,u_y(x,y) - u(x,y)\,v_y(x,y)}{v^2(x,y)} \right\rangle$$

$$= \frac{1}{v^2(x,y)}\Big(\langle v(x,y)\,u_x(x,y),\ v(x,y)\,u_y(x,y) \rangle - \langle u(x,y)\,v_x(x,y),\ u(x,y)\,v_y(x,y) \rangle \Big)$$

$$= \frac{1}{v^2(x,y)}\Big(v(x,y)\,\nabla u(x,y) - u(x,y)\,\nabla v(x,y) \Big)$$

d) If $f(x,y) = u^n(x,y)$

then

$$\nabla f(x,y) = \langle n\,u^{n-1}(x,y)\,u_x(x,y),\ n\,u^{n-1}(x,y)\,u_y(x,y) \rangle$$

$$= n\,u^{n-1}(x,y)\,\nabla u(x,y)$$

10. The points on $f(x,y,z) = x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to $z = x + y$ or $x + y - z = 0$ are those where

$$\nabla f(x_0, y_0, z_0) = \alpha \underbrace{\langle 1, 1, -1 \rangle}_{\substack{\text{normal} \\ \text{vector} \\ \text{of} \\ x+y-z}}$$

So

$$\langle 2x_0, -2y_0, -2z_0 \rangle = \alpha \langle 1, 1, -1 \rangle = \langle \alpha, \alpha, -\alpha \rangle$$

So, $x_0 = \frac{\alpha}{2}$, $y_0 = \frac{-\alpha}{2}$ and $z_0 = \frac{\alpha}{2}$

The points (x_0, y_0, z_0) must lie on $x^2 - y^2 - z^2 = 1$

so

$$\left(\frac{\alpha}{2}\right)^2 - \left(\frac{-\alpha}{2}\right)^2 - \left(\frac{\alpha}{2}\right)^2 = 1$$

$$-\frac{\alpha^2}{4} = 1 \Rightarrow \underbrace{\alpha^2 = -4}$$

Since we need x_0, y_0, z_0 to be real numbers, there is no value of α, that satisfy the equation above, and therefore, we can conclude that there is no such point.