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Section: 51

MATH 243 - Quiz 2

September 13, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Determine whether the vectors $\vec{A} = -\hat{\imath} + \hat{\jmath}$, $\vec{B} = -\hat{\imath} - \hat{\jmath} - 2\hat{k}$, and $\vec{C} = 2\hat{\imath} + 2\hat{k}$ form a triangle. If so, determine whether it is a right triangle.

   Solution:

   First, we need to make sure that the given vectors form a triangle. This will happen if the sum of two of them is equal to the third vector, or if the sum of all three vectors is equal to the zero vector.

   In this case, $\vec{A} + \vec{B} + \vec{C} = \vec{0}$.

   This triangle would be a right triangle if the angle between two vectors is equal to $\pi/2$, or equivalently, if their dot product is equal to zero.

   In this case, $\vec{A} \cdot \vec{B} = (-1)(-1) + (1)(-1) + (0)(-2) = 0$. Therefore, the vectors $\vec{A}$ and $\vec{B}$ are perpendicular and the triangle formed by $\vec{A}$, $\vec{B}$, and $\vec{C}$ is a right triangle.

2. (25 pts) What is the value of the triple product $\vec{A} \cdot (\vec{A} \times \vec{B})$? Why?

   Solution:

   The value of $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$ because the result of the cross product $\vec{A} \times \vec{B}$ is a vector perpendicular to both $\vec{A}$ and $\vec{B}$, and since the dot product of two perpendicular vectors is zero, the dot product of $\vec{A}$ and $\vec{A} \times \vec{B}$ is equal to zero.

3. (25 pts) Find the shortest distance from $P(6, -4)$ to the line $y = 2x - 3$.

   Solution:

If you recall, we solved a similar problem in homework 2 (problem 6). We found then that the distance d between a point and a line in $\mathbb{R}^2$ can be computed using $d = \frac{||\vec{AB} \times \vec{AP}||}{||\vec{AB}||}$, where $\vec{AB}$ is a vector between two points A and B on the line, and $\vec{AP}$ is the vector from point A to the point P we are interested in finding the distance to from the line.

Let A be the point on the line when $x = 0$ and B the point on the line when $x = 1$. So, $A(0, -3)$ and $B(1, -1)$. Now, $\vec{AB} = \langle 1, 2, 0 \rangle$ and $\vec{AP} = \langle 6, -1, 0 \rangle$. Computing the cross product we obtain:

$$\vec{AB} \times \vec{AP} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 6 & -1 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (-1 - 12)\hat{k} = \langle 0, 0, -13 \rangle.$$

Then, $||\vec{AB} \times \vec{AP}|| = 13$ and $||\vec{AB}|| = \sqrt{5}$. Thus, finally we can say that $d = \frac{13}{\sqrt{5}}$.

4. (25 pts) Find the area of a triangle with vertices at $A(3, -1, 2)$, $B(1, -1, -3)$, and $C(4, -3, 1)$

Solution:

We first define vectors to represent two sides of the triangle. So, $\vec{AB} = \langle -2, 0, -5 \rangle$, and $\vec{AC} = \langle 1, -2, -1 \rangle$. The area of the triangle is equal to $a = \frac{||\vec{AB} \times \vec{AC}||}{2}$. So

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix} = -10\hat{i} - 7\hat{j} + 4\hat{k} = \langle -10, -7, 4 \rangle.$$

Then, $||\vec{AB} \times \vec{AC}|| = \sqrt{(-10)^2 + (-7)^2 + (4)^2} = \sqrt{165}$, and $a = \frac{\sqrt{165}}{2}$.

Bonus (10 pts): Simplify $\vec{A} \cdot (2\vec{A} + \vec{B}) \times \vec{C}$

Solution:

Using the properties of the dot and cross products, we have:

$\vec{A} \cdot (2\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \cdot (2\vec{A} \times \vec{C} + \vec{B} \times \vec{C}) = \vec{A} \cdot (2\vec{A} \times \vec{C}) + \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{A} \cdot (2(\vec{A} \times \vec{C})) + \vec{A} \cdot (\vec{B} \times \vec{C}) = 2\vec{A} \cdot (\vec{A} \times \vec{C}) + \vec{A} \cdot (\vec{B} \times \vec{C})$.

However, $\vec{A} \cdot (\vec{A} \times \vec{C}) = 0$ (see problem 2, above). Thus,

$\vec{A} \cdot (2\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \cdot (\vec{B} \times \vec{C})$.