Name: Marco A. Montes de Oca — MATH 243 - Quiz 4

Section: 51 — October 11, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible. (If your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Find the curvature of $\vec{r}(t) = \langle 1-t, 3+5t, t\rangle$.

   Solution: By inspecting the vector function, we can see that the component functions are linear. We can conclude therefore, that the function represents a line in space and that its curvature is zero.

   Alternative: $\vec{r}'(t) = \langle -1, 5, 1\rangle$, $\vec{r}''(t) = \langle 0,0,0\rangle$. Therefore, $||\vec{r}'(t)\times\vec{r}''(t)|| = 0$ and $\kappa(t) = \dfrac{||\vec{r}'(t)\times\vec{r}''(t)||}{||\vec{r}'(t)||^3} = 0$.

2. (25 pts) Find the tangential and normal components of the acceleration of a particle whose position function is $\vec{r}(t) = \langle 3t^2, 3t-t^3, 3\rangle$.

   Solution: The velocity of the particle is given by $\vec{v}(t) = \vec{r}'(t) = \langle 6t, 3-3t^2, 0\rangle$. Its acceleration is given by $\vec{a}(t) = \vec{r}''(t) = \langle 6, -6t, 0\rangle$.

   We need to find the scalars a_T and a_N such that $\vec{a}(t) = a_T\vec{T}(t) + a_N\vec{N}(t)$, where $\vec{T}(t)$ and $\vec{N}(t)$ are the unit tangent and the principal unit normal vectors, respectively.

   By definition, $a_T = \vec{a}(t)\cdot\vec{T}(t)$, so we need to find $\vec{T}(t)$ first. So, $||\vec{r}'(t)|| = \sqrt{(6t)^2 + (3-3t^2)^2} = \sqrt{36t^2+9-18t^2+9t^4} = \sqrt{9t^4+18t^2+9} = \sqrt{(3t^2+3)^2} = |3t^2+3| = 3t^2+3$ (because t^2 is always greater than zero).

   Therefore, $\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} = \left\langle \frac{6t}{3t^2+3}, \frac{3-3t^2}{3t^2+3}, 0\right\rangle$.

   So, $a_T = \vec{a}(t)\cdot\vec{T}(t) = \langle 6, -6t, 0\rangle \cdot \left\langle \frac{6t}{3t^2+3}, \frac{3-3t^2}{3t^2+3}, 0\right\rangle = \frac{36t}{3t^2+3} - \frac{6t(3-3t^2)}{3t^2+3} = \frac{36t}{3t^2+3} - \frac{18t}{3t^2+3} + \frac{18t^3}{3t^2+3} = \frac{18t}{3t^2+3} + \frac{18t^3}{3t^2+3} = \frac{6t(3t^2+3)}{3t^2+3} = 6t$. Thus, $a_T = 6t$.

   The normal component of the acceleration can be found by the Pythagorean theorem, by noting that $||\vec{a}(t)||^2 = a_T^2 + a_N^2$, so $a_N^2 = ||\vec{a}(t)||^2 - a_T^2 = (36+36t^2) - 36t^2 = 36$, and therefore, $a_N = \sqrt{36} = 6$.

3. (25 pts) Find all the points on the graph of $y = 1 - x^3$ such that the curvature is zero.

Solution: The parametrization of this function is $\vec{r}(t) = \langle t, 1 - t^3, 0\rangle$. Therefore its curvature is $\kappa(t) = \dfrac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3}$.

The vector $\vec{r}'(t) - \langle 1, -3t^2, 0\rangle$, $\vec{r}''(t) = \langle 0, -6t, 0\rangle$. So $\vec{r}'(t) \times \vec{r}''(t) = \langle 0, 0, -6t\rangle$ and $||\vec{r}'(t) \times \vec{r}''(t)|| = |-6t| = 6|t|$. The curvature of $y = 1-x^3$ is thus $\kappa(t) = \dfrac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3} = \dfrac{6|t|}{(1+9t^4)^{3/2}}$.

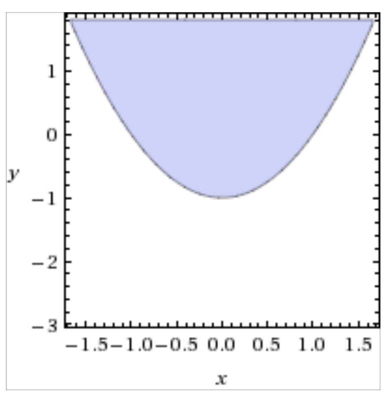
It is apparent that the curvature of this curve is zero only when $t = 0$; thus, the only point at which the curvature is zero is at $(0, 1)$.

Alternative: In homework # 6, you showed that the graph of a function has curvature zero at its inflection points, that is, when $f''(x) = 0$. In this problem this means that $y = 1 - x^3$ will have curvature zero when $-6x = 0$, that is, when $x = 0$. The point at which the curvature is zero is then $(0, 1)$.

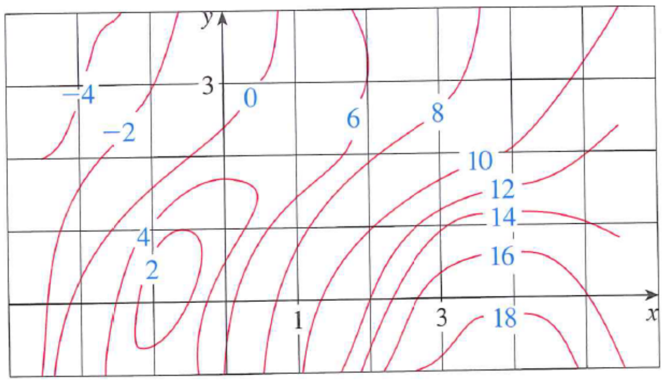
4. (25 pts) The domain of a function is the set of all pairs (x, y) for which the function is a well-defined real number. Find and sketch the domain of the function $f(x, y) = \sqrt{1 + y - x^2}$.

Solution: The function $f(x, y) = \sqrt{1 + y - x^2}$ returns a real number only when $1+y-x^2 \geq 0$, which means that the region in the xy-plane that satisfies this inequality is $y \geq x^2 - 1$.

A sketch of this region is



Bonus (10 pts): Using the contour plot shown below, estimate the value of $f(0, 0)$ and $f(-2, 2)$.



Solution: Since $(0,0)$ is located between the controul lines with values 4 and 6, it is reasonable to assume that $4 < f(0,0) < 6$. My personal estimate is that $f(0,0) = 5.8$. For $(-2,2)$, $-4 < f(-2,2) < -2$, so $f(-2,2) = -2.8$.