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Section: 51 November 8, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Find a vector perpendicular to the level curve $f(x, y) = xy = -3$ at $(-1, 3)$.

   Solution: One of the properties of the gradient vector of a function $f(x, y)$ at a point (x_0, y_0) is that it is perpendicular to the level curve that passes through (x_0, y_0).

   For this problem, the gradient is $\nabla f(x, y) = \langle y, x\rangle$, and at $(-1, 3)$ it is $\nabla f(-1, 3) = \langle 3, -1\rangle$. Thus, the vector $\langle 3, -1\rangle$ is a vector perpendicular to the level curve $f(x, y) = -3$.

2. (25 pts) Find and classify all the critical points of $f(x, y) = 4 + x^3 + y^3 - 3xy$.

   Solution: The critical points of $f(x, y)$ satisfy $\nabla f(x, y) = \vec{0}$.

   If $\nabla f(x, y) = \langle 3x^2 - 3y, 3y^2 - 3x\rangle = \langle 0, 0\rangle$, then $x^2 - y = 0$ (1) and $y^2 - x = 0$ (2). Substituting (1) in (2): $(y^2)^2 - y = y^4 - y = y(y^3 - 1) = 0$. This means that the critical points are $(0, 0)$ and $(1, 1)$.

   To know whether at these points the function has local maxima, local minima or saddle points, we need to compute the determinant of the Hessian matrix of f. The components of the Hessian matrix are $f_{xx}(x, y) = 6x$, $f_{yy}(x, y) = 6y$ and $f_{xy}(x, y) = f_{yx}(x, y) = -3$. Its determinant is $\det(H)(x, y) = (6x)(6y) - (-3)^2 = 36xy - 9$.

   At $(0, 0)$, $\det(H)(0, 0) = -9 < 0$. Therefore, at $(0, 0)$, the function has a saddle point.

   At $(1, 1)$, $\det(H)(1, 1) = 36 - 9 = 27 > 0$ and $f_{xx}(1, 1) = 6 > 0$. Therefore, at $(1, 1)$, the function has a local minimum.

3. (25 pts) Two objects are traveling along the paths given by the following parametric equations:

$$x_1 = 2t \text{ and } y_1 = 2 - 4t^2 \qquad\qquad \text{First object}$$
$$x_2 = t - 1 \text{ and } y_2 = t \qquad\qquad \text{Second object}$$

At what rate is the distance between the two objects changing when $t = 1$? (Hint: Use the Chain Rule)

Solution: The distance between the two objects is given by $w = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. This is a function of four independent variables that depend on t, therefore:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial w}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial w}{\partial y_2}\frac{dy_2}{dt}.$$

$$\frac{\partial w}{\partial x_1} = \frac{1}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2)^{-1/2}(2(x_1 - x_2)) = \frac{x_1 - x_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}.$$

$$\frac{\partial w}{\partial x_2} = \frac{1}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2)^{-1/2}(2(x_1 - x_2)(-1)) = \frac{x_2 - x_1}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}.$$

$$\frac{\partial w}{\partial y_1} = \frac{1}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2)^{-1/2}(2(y_1 - y_2)) = \frac{y_1 - y_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}.$$

$$\frac{\partial w}{\partial y_2} = \frac{1}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2)^{-1/2}(2(y_1 - y_2)(-1)) = \frac{y_2 - y_1}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}.$$

$$\frac{dx_1}{dt} = 2.$$
$$\frac{dx_2}{dt} = 1.$$
$$\frac{dy_1}{dt} = -8t.$$
$$\frac{dy_2}{dt} = 1.$$

At $t = 1$, $x_1 = 2$, $x_2 = 0$, $y_1 = -2$, $y_2 = 1$. So at $t = 1$, $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(2 - 0)^2 + (-2 - 1)^2} = \sqrt{4 + 9} = \sqrt{13}$. So

$$\left.\frac{dw}{dt}\right|_{t=1} = \frac{(2-0)}{\sqrt{13}}(2)+\frac{(0-2)}{\sqrt{13}}(1)+\frac{(-2-1)}{\sqrt{13}}(-8)+\frac{(1-(-2))}{\sqrt{13}}(1) = \frac{4}{\sqrt{13}}-\frac{2}{\sqrt{13}}+\frac{24}{\sqrt{13}}+\frac{3}{\sqrt{13}}.$$

Therefore, $\left.\frac{dw}{dt}\right|_{t=1} = \frac{29}{\sqrt{13}}$.

4. (25 pts) Show that if f is a differentiable function such that $\nabla f(x_0, y_0) = \vec{0}$, then the tangent plane at (x_0, y_0) is horizontal.

Solution: A tangent plane to f at (x_0, y_0) is given by $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. If $\nabla f(x_0, y_0) = \vec{0}$, then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Under these circumstances, the tangent plane to f at (x_0, y_0) is given by $z - z_0 = 0$ or $z = z_0$, which means that the plane is horizontal.

Bonus (10 pts): Find the minimum distance from the point $(2, 2, 0)$ to the surface $z = x^2 + y^2$.

Solution: The minimization problem is:

Minimize $f(x, y, z) = (x-2)^2 + (y-2)^2 + z^2$, subject to $g(x, y, z) = x^2 + y^2 - z = 0$.

Using Lagrange multipliers, we have $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.

$\nabla f(x, y, z) = \langle 2(x-2), 2(y-2), 2z \rangle = \lambda \langle 2x, 2y, -1 \rangle$. The system of equation is:

$2x - 4 = 2\lambda x$ (1)
$2y - 4 = 2\lambda y$ (2)
$2z = -\lambda$ (3)
$x^2 + y^2 - z = 0$ (4)

From (1), (2), and (3): $x = \frac{2}{1-\lambda}$, $y = \frac{2}{1-\lambda}$, $z = -\frac{\lambda}{2}$. Substituting in (4):

$(\frac{2}{1-\lambda})^2 + (\frac{2}{1-\lambda})^2 + \frac{\lambda}{2} = 0$
$2(\frac{2}{1-\lambda})^2 = -\frac{\lambda}{2}$
$16 = -((1-\lambda)^2)(\lambda) = -(1 - 2\lambda + \lambda^2)(\lambda) = -\lambda + 2\lambda^2 - \lambda^3$
$\lambda^3 - 2\lambda^2 + \lambda + 16 = 0$

The only real solution of this equation is approximately $\lambda = -1.901$, therefore,

$x = y = 0.689$, $z = 0.9505$, and $d = \sqrt{f(x, y, z)} = 2.083$