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Section: 50 & 51 — December 3, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Determine whether $\vec{F} = \langle 3x^2 + y^2, 2xy - 3y^2 \rangle$ is a conservative field. If it is, find f such that $\vec{F} = \nabla f$.

   Solution: $\vec{F}$ is conservative if $\nabla \times \vec{F} = \vec{0}$.

   $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 + y^2 & 2xy - 3y^2 & 0 \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (2y - 2y)\hat{k} = \vec{0}$. Therefore, $\vec{F}$ is conservative.

   If $f_x = 3x^2 + y^2$, then $f = x^3 + y^2 x + C(y)$. So, $f_y = 2xy + C'(y)$. By comparison with the given field we notice that $C'(y) = -3y^2$. Therefore, $C(y) = -y^3 + K$. We conclude therefore that $f(x,y)$ such that $\vec{F} = \nabla f$ is $f(x,y) = x^3 + y^2 x - y^3 + K$.

2. (25 pts) Evaluate $\int_C x^2 \, ds$, where C is a circle (defined in a counterclockwise direction) of radius 4.

   Solution: C: $\vec{r}(t) = \langle 4\cos t, 4\sin t \rangle$. Then $\vec{r}\,'(t) = \langle -4\sin t, 4\cos t \rangle$, and $||\vec{r}(t)|| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = \sqrt{16} = 4$.

   Then $\displaystyle\int_C x^2 \, ds = \int_0^{2\pi} (4\cos t)^2 ||\vec{r}(t)|| dt = \int_0^{2\pi} (4\cos t)^2 (4) dt$

   $\displaystyle\int_0^{2\pi} (4\cos t)^2 (4) dt = 64 \int_0^{2\pi} \cos^2 t \, dt$. Using $\cos^2 t = \frac{1}{2}(1 + \cos(2t))$, we get

   $64 \int_0^{2\pi} \cos^2 t \, dt = 64\pi$.

3. (25 pts) Evaluate $\int_C xy\,dx + y\,dy$, where C is the straight line segment from $(0,0)$ to $(1,1)$.

Solution: The equation of the line that passes through $(0,0)$ and $(1,1)$ is $x = y$. Substituting y and dy in the integral:

$$\int_C xy\,dx + y\,dy = \int_0^1 x(x)\,dx + x\,dx = \int_0^1 (x^2 + x)\,dx = \left.\frac{x^3}{3} + \frac{x^2}{2}\right]_0^2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

4. (25 pts) Determine whether it is possible to use the Fundamental Theorem of Calculus for Line Integrals in the following integral. If it is, use it. $\int_C 3x^2y\,dx + (x^3 + \cos(y))\,dy$, where C is given by $\vec{r}(t) = \langle \cos(t)\sin(t), \sin(4t)\rangle$, $0 \le t \le \pi$.

Solution: We can use the Fundamental Theorem of Calculus for Line Integrals if $\vec{F}$ is conservative. $\vec{F}$ is conservative if $\nabla \times \vec{F} = \vec{0}$. So

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & x^3 + \cos(y) & 0 \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (3x^2 - 3x^2)\hat{k} = \vec{0},$$ so yes, we can use the Fundamental Theorem of Calculus for Line Integrals.

If $f_x = 3x^2y$ then $f = x^3y + C(y)$. Then $f_y = x^3 + C'(y)$ and by comparison with the field, we notice that $C'(y) = \cos y$. Therefore $C(y) = \sin y + K$. The potential function is therefore $f(x,y) = x^3y + \sin y + K$.

The Fundamental Theorem of Calculus for Line Integrals says that if $\vec{F} = \nabla f$ for some f, then

$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$, where $\vec{r}(b)$ is where the path ends and $\vec{r}(a)$ is where the path starts.

In our case, $\vec{r}(\pi) = \langle 0,0\rangle$ (final point), and $\vec{r}(0) = \langle 0,0\rangle$ (starting point). We notice that they are the same point. Therefore, by the Fundamental Theorem of Calculus for Line Integrals:

$$\int_C 3x^2y\,dx + (x^3 + \cos(y))\,dy = 0, \text{ where } C \text{ is given by } \vec{r}(t) = \langle \cos(t)\sin(t), \sin(4t)\rangle,\ 0 \le t \le \pi$$

Bonus (10 pts): Let $\vec{F} = \langle P, Q, R \rangle$. If $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$, what would $\nabla \cdot \vec{F}$ be equal to? [$\nabla \cdot \vec{F}$ is called the *divergence* of $\vec{F}$.]

Solution: $\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$