University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
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Solution Exam I

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Problems

1. [20 points in total] Find an equation for the line through $(9, -100, 1)$ and $(-1, 2, 3)$.

   Solution: The direction vector of the line is $\overrightarrow{v} = \langle -1-(9), 2-(-100), 3-(1)\rangle = \langle -10, 102, 2\rangle$. Thus, an equation for this line is:

   $\overrightarrow{r}(t) = \langle 9, -100, 1\rangle + t\langle -10, 102, 2\rangle = \langle 9-10t, -100+102t, 1+2t\rangle$.

   Other solutions are possible.

2. [20 points in total] Find an equation of the plane through $(2, 0, -1)$ and parallel to $x + y - 2z = -10000$.

   Solution: If two planes are parallel, they have the parallel normal vectors. Therefore, an equation for the sought plane is:

   $1(x-2) + 1(y-0) - 2(z+1) = x + y - 2z - 2 - 2 = x + y - 2z - 4 = 0$.

3. [20 points in total] Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $x - y + z = 2$.

Solution: The projection of the intersection curve onto the xy-plane is simply $x^2 + y^2 = 16$, which can be parametrized as $x = 4\cos t$, $y = 4\sin t$.

Now, from the equation of the plane, we see that $z = 2 - x + y = 2 - 4\cos t + 4\sin t$. Therefore, the vector function representing the intersection curve is:

$\overrightarrow{r}(t) = \langle 4\cos t, 4\sin t, 2 - 4\cos t + 4\sin t\rangle$.

4. [20 points in total] Find the length of the curve $\overrightarrow{r}(t) = \langle 2t, \frac{4}{3}t^{3/2}, \frac{1}{2}t^2\rangle$, $0 \le t \le 1$.

Solution: The length of the curve is given by $\int_0^1 ||\overrightarrow{r}\,'(t)||\,dt$. So, let's find $||\overrightarrow{r}\,'(t)||$:

$\overrightarrow{r}\,'(t) = \langle 2, 2t^{1/2}, t\rangle$, therefore $||\overrightarrow{r}\,'(t)|| = \sqrt{4 + 4t + t^2} = \sqrt{(2+t)^2} = |2 + t| = 2 + t$, because $t \ge 0$.

Thus, $\int_0^1 ||\overrightarrow{r}\,'(t)||\,dt = \int_0^1 (2 + t)\,dt = 2t + \frac{t^2}{2}|_0^1 = 2 + \frac{1}{2} = \frac{5}{2}$

5. [20 points in total] Show that the circular helix $\overrightarrow{r}(t) = \langle a\cos t, a\sin t, bt\rangle$, where a and b are positive constants, has constant curvature.

Solution: Let's use $\kappa(t) = \frac{||\overrightarrow{r}\,'(t)\times\overrightarrow{r}\,''(t)||}{||\overrightarrow{r}\,'(t)||^3}$.

$\overrightarrow{r}\,'(t) = \langle -a\sin t, a\cos t, b\rangle$, $\overrightarrow{r}\,''(t) = \langle -a\cos t, -a\sin t, 0\rangle$. Therefore,
$\overrightarrow{r}\,'(t) \times \overrightarrow{r}\,''(t) = \langle ab\sin t, -ab\cos t, a^2\rangle$, which means that
$||\overrightarrow{r}\,'(t) \times \overrightarrow{r}\,''(t)|| = \sqrt{a^2b^2\sin^2 t + a^2b^2\cos^2 t + a^4} = \sqrt{a^2b^2 + a^4} = a\sqrt{b^2 + a^2}$.

Now, $||\overrightarrow{r}\,'(t)||^3 = (a^2 + b^2)^{3/2} = (\sqrt{a^2 + b^2})^3$. Therefore

$\kappa(t) = \frac{||\overrightarrow{r}\,'(t)\times\overrightarrow{r}\,''(t)||}{||\overrightarrow{r}\,'(t)||^3} = \frac{a\sqrt{b^2+a^2}}{(\sqrt{a^2+b^2})^3} = \frac{a}{(\sqrt{a^2+b^2})^2} = \frac{a}{a^2+b^2}$.

Since $\kappa(t)$ does not depend on t, the curvature is constant.

[Bonus: 10 points] Find a vector tangent to the curve $\overrightarrow{r}(t) = \left\langle \int_0^t \sin\left(\frac{\pi\theta^2}{2}\right)\,d\theta, \int_0^t \cos\left(\frac{\pi\theta^2}{2}\right)\,d\theta \right\rangle$ at $t = 1$.

Solution: A vector tangent to $\overrightarrow{r}(t)$ is $\overrightarrow{r}\,'(t)$, so $\overrightarrow{r}\,'(t) = \left\langle \frac{d}{dt}\int_0^t \sin\left(\frac{\pi\theta^2}{2}\right)\,d\theta, \frac{d}{dt}\int_0^t \cos\left(\frac{\pi\theta^2}{2}\right)\,d\theta \right\rangle$.

By the Fundamental Theorem of Calculus:

$\overrightarrow{r}\,'(t) = \langle \sin\left(\frac{\pi t^2}{2}\right), \cos\left(\frac{\pi t^2}{2}\right)\rangle$

At $t = 1$: $\overrightarrow{r}\,'(1) = \langle 1, 0\rangle$