University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
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Exam II

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Problems

1. [20 points in total] Find the velocity and position vectors of a particle that has an acceleration given by $\boldsymbol{a}(t) = \langle 4t+1, \cos t, e^t \rangle$, whose initial velocity and position vectors at $t = 0$ are $\boldsymbol{v}(0) = \langle 1, 1, 1 \rangle$ and $\boldsymbol{r}(0) = \langle 5, 1, 0 \rangle$, respectively.

   Solution: If $\boldsymbol{a}(t) = \langle 4t + 1, \cos t, e^t \rangle$, then $\boldsymbol{v}(t) = \int \boldsymbol{a}(t)\, dt = \langle 2t^2 + t, \sin t, e^t \rangle + \boldsymbol{C}$.

   Since $\boldsymbol{v}(0) = \langle 1, 1, 1 \rangle$, $\boldsymbol{C} = \langle 1, 1, 0 \rangle$. Therefore, $\boldsymbol{v}(t) = \langle 2t^2 + t + 1, \sin t + 1, e^t \rangle$.

   Now, $\boldsymbol{r}(t) = \int \boldsymbol{v}(t)\, dt = \langle \frac{2}{3}t^3 + \frac{1}{2}t^2 + t, -\cos t + t, e^t \rangle + \boldsymbol{D}$.

   Since $\boldsymbol{r}(0) = \langle 5, 1, 0 \rangle$, then $\boldsymbol{D} = \langle 5, 2, -1 \rangle$. Thus, $\boldsymbol{r}(t) = \langle \frac{2}{3}t^3 + \frac{1}{2}t^2 + t + 5, -\cos t + t + 2, e^t - 1 \rangle$.

2. [20 points in total] Find an equation of the tangent plane to $z = \frac{xy}{xy+1}$ at $(1, 1)$.

   Solution: $f_x(x, y) = \frac{(xy+1)y - xy(y)}{(xy+1)^2} = \frac{y}{(xy+1)^2}$. Similarly, $f_y(x, y) = \frac{(xy+1)x - xy(x)}{(xy+1)^2} = \frac{x}{(xy+1)^2}$. Thus, $f_x(1, 1) = f_y(1, 1) = \frac{1}{4}$.

The equation of the tangent plane is thus:

$z - \frac{1}{2} = \frac{1}{4}(x-1) + \frac{1}{4}(y-1)$, or $x + y - 4z = 0$.

3. [20 points in total] Find the directional derivative of $f(x,y) = \ln(x^2y)$ at the point $(1,1)$ in the direction of the vector $\boldsymbol{v} = \hat{\imath} + \hat{\jmath}$.

   Solution: $\hat{v} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$. Thus, $D_{\hat{v}}f(1,1) = \nabla f(1,1) \cdot \hat{v}$. Since $\nabla f(x,y) = \langle \frac{2xy}{x^2y}, \frac{x^2}{x^2y} \rangle = \langle \frac{2}{x}, \frac{1}{y} \rangle$, then $\nabla f(1,1) = \langle 2, 1 \rangle$.

   $D_{\hat{v}}f(1,1) = \langle 2,1 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$.

4. [20 points in total] Find and classify all the critical points of $f(x,y) = x^4 - 2x^2 - 2y^2 + 2xy$.

   Solution: $\nabla f(x,y) = \langle 4x^3 - 4x + 2y, -4y + 2x \rangle$. If $\nabla f(x,y) = \mathbf{0}$, then
   $4x^3 - 4x + 2y = 0$ (1) and $-4y + 2x = 0$ (2).

   From (2), $x = 2y$. Substituting x in (1): $4(2y)^3 - 4(2y) + 2y = 32y^3 - 6y = 2y(16y^2 - 3) = 0$. This means that either $y = 0$, or $y = \pm\frac{\sqrt{3}}{4}$.

   So the critical points of the function are: $(0,0)$, $(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4})$, and $(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{4})$.

   $f_{xx} = 12x^2 - 4$ and the determinant of the Hessian of f is $D = 12 - 48x^2$. At $(0,0)$, $f_{xx} = -4$ and $D = 12 > 0$. Thus, $(0,0)$ is local maximizer of f. At $(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4})$, $D = -24 < 0$. Therefore, $(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4})$ is a saddle point. At $(\frac{-\sqrt{3}}{2}, -\frac{\sqrt{3}}{4})$, $D = -24 < 0$. Therefore, $(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{4})$ is a saddle point.

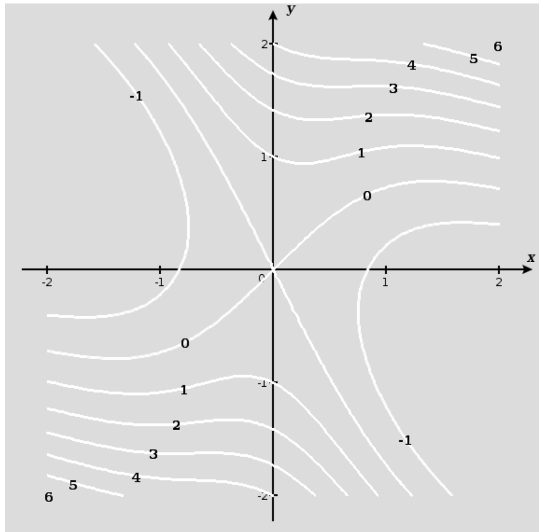
5. [20 points in total] Use Lagrange multipliers to show that the rectangle with maximum area that has a given perimeter p is a square.

   Solution: Let b and h represent the base and height of the rectangle, respectively. Then, the area $A(b,h) = bl$ and the perimeter $p = 2b + 2h$.

   The optimization problem is to maximize $A(b,h)$ subject to $p = 2b + 2l$. Thus, the Lagrange multiplier equation is $\nabla A(b,h) = \lambda \nabla(2b + 2l)$.

   Thus, $\langle l, b \rangle = \lambda \langle 2, 2 \rangle$. This implies that the solution of the problem must satisfy $l = 2\lambda$ and $b = 2\lambda$, which means that b must be equal to l, and thus the rectangle is actually a square.

[Bonus: 10 points] Using the contour plot below, estimate the sign of $\frac{\partial f}{\partial y}$ at $(1,1)$. Explain your choice. (Without a satisfactory explanation, no credit will be given.)



Solution: $\frac{\partial f}{\partial y} > 0$ at $(1, 1)$ because f increases as y increases at $(1, 1)$.