University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
Instructor: Dr. Marco A. Montes de Oca
Spring 2013

Exam III

Name: ______________________________ **Section:** 51

May 6, 2013

Question	1	2	3	4	5	Bonus	Total
Points							

Instructions

- The exam consists of **five** problems for a total of 100 points, plus a bonus problem for 10 extra points.
- Read very carefully each problem before working on it.
- Partial credit will not be given if appropriate work is not shown.
- If you get stuck on a problem, skip it and come back to it if you have extra time at the end.
- Answer questions in the space provided. If you need more space for an answer, continue your answer on the back of the page, or/and use the margins of the test pages.
- Carefully work out each problem and clearly indicate your final answer to any problem.
- You may **not** use calculators, dictionaries, notes, or any other kinds of aids.
- **DISHONESTY WILL NOT BE TOLERATED.**

Problems

1. [20 points in total] Change into spherical coordinates: $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dz\,dy\,dx$. DO NOT EVALUATE.

   Solution:

   From the limits of integration wrt z: Lower limit: $\rho\cos\phi = \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} = \rho\sin\phi$. Which implies $\tan\phi = 1$, which means $\phi = \frac{\pi}{4}$.

   Upper limit: $\rho\cos\phi = \sqrt{8 - (\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2} = \sqrt{8 - \rho^2\sin^2\phi}$. Squaring both sides and rearranging terms: $\rho^2 = 8$, so $\rho = \sqrt{8}$.

   From this analysis, we may conclude that $0 \le \rho \le \sqrt{8}$, and $0 \le \phi \le \pi/4$.

   Since the region of integration in the xy-plane is simply a circle of radius 2 (we can verify that $2 = \sqrt{8}\sin(\pi/4)$), we can conclude:

   $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dz\,dy\,dx = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

2. [20 points in total] Evaluate the following line integral: $\int_C (x-1)\,ds$, where C is the arc of the parabola $y = (x-1)^2$ from $(0,1)$ to $(1,0)$.

   Solution:

   $C : \boldsymbol{r}(t) = \langle t, (t-1)^2\rangle$, so $\boldsymbol{r}'(t) = \langle 1, 2(t-1)\rangle$ and $||\boldsymbol{r}'(t)|| = \sqrt{1 + 4(t-1)^2}$. Thus,

   $\int_C (x-1)\,ds = \int_0^1 (t-1)\sqrt{1+4(t-1)^2}\,dt$. Setting $u = 1 + 4(t-1)^2$, $du = 8(t-1)$. Therefore

   $$\frac{1}{8}\int_5^1 \sqrt{u}\,du = \frac{1}{8}\left[\frac{2}{3}u^{3/2}\right]_5^1 = \frac{1}{12}(1^{3/2} - 5^{3/2}) = \frac{1}{12}(1 - 5^{3/2}).$$

3. [20 points in total] Show that $\boldsymbol{F} = \langle 4x^3y^2 - 2xy^3, 2x^4y - 3x^2y^2 + 4y^3\rangle$ is conservative and use this fact to evaluate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$, where $C : \boldsymbol{r}(t) = \langle t + \sin(\pi t), 2t + \cos(\pi t)\rangle$, $0 \le t \le 1$.

   Solution: Since $Q_x = 8x^3y - 6xy^2$ and $P_y = 8x^3y - 6xy^2$, $Q_x - P_y = 0$, which shows that $\boldsymbol{F}$ is a gradient field, and therefore conservative.

   We need to find $f(x,y)$ such that $\nabla f(x,y) = \boldsymbol{F}$. Therefore, $f_c = \int(4x^3y^2 - 2xy^3)\,dx = x^4y^2 - x^2y^3 + C(y)$. Now, $\partial f_c/\partial y = 2x^4y - 3x^2y^2 + C'(y) = 2x^4y - 3x^2y^2 + 4y^3$. Therefore, $C'(y) = 4y^3$ and $C(y) = y^4 + K$. We conclude, therefore that $f(x,y) = x^4y^2 - x^2y^3 + y^4 + K$.

By the FTC4LIs, we have that $\int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r}(t_{\text{final}})) - f(\boldsymbol{r}(t_{\text{initial}})) = f(\boldsymbol{r}(1)) - f(\boldsymbol{r}(0)) = f(1,1) - f(0,1) = 1 - 1 = 0$.

4. [20 points in total] Use Green's Theorem to evaluate $\oint_C \sqrt{1+\arctan(x^5)}\,dx + 2xy\,dy$, where C is the triangle with vertices $(0,0)$, $(1,2)$, $(0,3)$.

   Solution: According to Green's Theorem: $\oint_C \sqrt{1+\arctan(x^5)}\,dx + 2xy\,dy = \iint_D (Q_x - P_y)\,dA = \iint_D 2y\,dA$, where D is the region bounded by $y = 2x$, $y = -x+3$ and the y-axis.

   Therefore, $\iint_D 2y\,dA = \int_0^1 \int_{2x}^{-x+3} 2y\,dy\,dx = \int_0^1 y^2\Big]_{2x}^{-x+3}\,dx = \int_0^1 ((-x+3)^2 - (2x)^2)\,dx = \int_0^1 (x^2 - 6x + 9 - 4x^2)\,dx = \int_0^1 (-3x^2 - 6x + 9)\,dx = -x^3 - 3x^2 + 9x\Big]_0^1 = -1 - 3 + 9 = 5$.

5. [20 points in total] Evaluate $\oint_C -y\,dx$ where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation, using your favorite method, that is, from scratch, using the Fundamental Theorem of Calculus for Line Integrals (if appropriate), or Green's Theorem (if appropriate).

   Solution: Since C is closed, we can use Green's Theorem. Then, $\oint_C -y\,dx = \int_D (0 - (-1))\,dA = \iint_D dA$. Since D is the disk centered at the origin with radius 2, $\iint_D dA$ is simply the area of that disk. Therefore, $\oint_C -y\,dx = 4\pi$.

[Bonus: 10 points] Is it true that $\nabla \times (\boldsymbol{F} + \boldsymbol{G}) = \nabla \times \boldsymbol{F} + \nabla \times \boldsymbol{G}$? Justify your answer.

Solution: Yes, by the properties of derivatives and the cross product.