Math 243 - Section 51

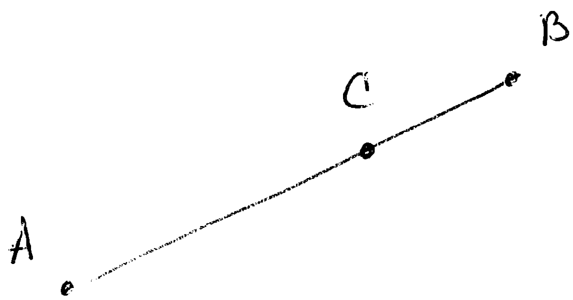
Spring 2013

Homework #1

1. Three points, A, B, and C lie on a straight line if

$$\|\overrightarrow{AB}\| + \|\overrightarrow{BC}\| = \|\overrightarrow{AC}\|$$

as shown below



$\overrightarrow{AB} = \langle 3-2, 7-4, -2-2 \rangle = \langle 1, 3, -4 \rangle$

$\overrightarrow{BC} = \langle 1-3, 3-7, 2-(-2) \rangle = \langle -2, -4, 4 \rangle$

$\overrightarrow{AC} = \langle 1-2, 3-4, 2-2 \rangle = \langle -1, -1, 0 \rangle$

$\|\overrightarrow{AB}\| = \sqrt{1^2 + 3^2 + (-4)^2} = \sqrt{1+9+16} = \sqrt{26}$

$\|\overrightarrow{BC}\| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4+16+16} = \sqrt{36} = 6$

$\|\overrightarrow{AC}\| = \sqrt{(-1)^2 + (-1)^2 + 0^2} = \sqrt{2}$

Since

$$\|\vec{AB}\| + \|\vec{BC}\| > \|\vec{AC}\|$$

We can conclude that the points A, B, C do not lie on a straight line, but rather form a triangle.

b) Using the same analysis:

$\vec{AB} = \langle 1-0, -2-(-5), 4-5 \rangle = \langle 1, 3, -1 \rangle$

$\vec{BC} = \langle 3-1, 4-(-2), 2-4 \rangle = \langle 2, 6, -2 \rangle$

$\vec{AC} = \langle 3-0, 4-(-5), 2-5 \rangle = \langle 3, 9, -3 \rangle$

$\|\vec{AB}\| = \sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{1+9+1} = \sqrt{11}$

$\|\vec{BC}\| = \sqrt{2^2 + 6^2 + (-2)^2} = \sqrt{4+36+4} = \sqrt{44} = 2\sqrt{11}$

$\|\vec{AC}\| = \sqrt{3^2 + 9^2 + (-3)^2} = \sqrt{9+81+9} = \sqrt{99} = 3\sqrt{11}$

Since

$$\|\vec{AB}\| + \|\vec{BC}\| = \sqrt{11} + 2\sqrt{11} = 3\sqrt{11} = \|\vec{AC}\|$$

we can conclude that A, B, and C lie on a straight line.

2. The midpoint between $(2,1,4)$ and $(4,3,10)$ is the sphere's center. We can compute it as $\left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right) = (3,2,7)$. The sphere's radius is the distance from $(3,2,7)$ to $(2,1,4)$ or $(4,3,10)$. Let this distance be denoted by r. Then,

$$r = \sqrt{(3-2)^2 + (2-1)^2 + (7-4)^2} = \sqrt{(1)^2 + (1)^2 + (3)^2}$$

$$= \sqrt{1+1+9} = \sqrt{11}$$

Then, the equation of the sphere with center $(3,2,7)$ and radius $\sqrt{11}$ is

$$(x-3)^2 + (y-2)^2 + (z-7)^2 = 11$$

3. The region between the yz-plane and the vertical plane $x=5$ can be represented by the inequalities

$$x \geq 0 \ \& \ x \leq 5$$

4.

a) $\vec{a} \cdot \vec{b} \cdot \vec{c}$ does not make sense because $\vec{a} \cdot \vec{b}$ or in general the dot product between two vectors is a scalar, so $\vec{a} \cdot \vec{b} \cdot \vec{c}$ implies the dot product between a scalar and a vector, which is not defined.

b) It is well defined.

$$(\vec{a} \cdot \vec{b})\vec{c} = (\langle 1,3,2\rangle \cdot \langle 0,-1,8\rangle)\langle 1,-2,0\rangle$$
$$= (1(0)+3(-1)+2(8))\langle 1,-2,0\rangle$$
$$= (0-3+16)\langle 1,-2,0\rangle = 13\langle 1,-2,0\rangle$$
$$= \langle 13,-26,0\rangle$$

c) It is well defined.

$$\|\vec{a}\|(\vec{b} \cdot \vec{c}) = \sqrt{1^2+3^2+2^2}\,(\langle 0,-1,8\rangle \cdot \langle 1,-2,0\rangle)$$
$$= \sqrt{1+9+4}\,(0+2+0) = \sqrt{14}\,(2) = 2\sqrt{14}$$

d) It is well defined.

$$\vec{a} \cdot (\vec{b}+\vec{c}) = \langle 1,3,2\rangle \cdot (\langle 0,-1,8\rangle + \langle 1,-2,0\rangle)$$
$$= \langle 1,3,2\rangle \cdot \langle 1,-3,8\rangle$$
$$= 1-9+16 = 8$$

e) It is not defined. A scalar cannot be added to a vector.

f) It is not defined. $\|\vec{a}\|$ is a scalar, which cannot be dot-multiplied

5. We know that for two non-zero vectors $\vec{a}$ and $\vec{b}$:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

where $0 \leq \theta \leq \pi$ is the angle between $\vec{a}$ and $\vec{b}$.

So, if $\vec{a} = \langle 3,4,0 \rangle$ and $\vec{b} = \langle 0,x,8 \rangle$, then

$$\langle 3,4,0 \rangle \cdot \langle 0,x,8 \rangle = 4x = \sqrt{3^2+4^2}\sqrt{x^2+8^2}\cos\theta$$

$$= \sqrt{9+16}\ \sqrt{x^2+64}\ (0.5) \text{ if } \theta = 60^\circ$$

$$4x = 2.5\sqrt{x^2+64} \quad \text{(squaring both sides)}$$

$$16x^2 = 6.25(x^2+64)$$

$$6.25x^2 - 16x^2 + 64 = 0$$

$$-9.75x^2 = -64$$

$$x^2 = \frac{-64}{-9.75} = 6.564$$

$$\Rightarrow x = \pm 2.562$$

6. If $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal, then

$$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = 0$$

So

$$\vec{u}\cdot(\vec{u}-\vec{v}) + \vec{v}\cdot(\vec{u}-\vec{v}) = 0$$

$$\vec{u}\cdot\vec{u} - \cancel{\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u}}^{\,0} - \vec{v}\cdot\vec{v} = 0$$

$$\|\vec{u}\|^2 - \|\vec{v}\|^2 = 0 \Rightarrow \|\vec{u}\| = \|\vec{v}\|$$

which is what we wanted to show.

7. $\vec{a} = \langle 1, 1, -2\rangle$, $\vec{b} = \langle 3, -2, 1\rangle$, $\vec{c} = \langle 0, 1, -5\rangle$

$$(2\vec{a}+3\vec{b})\cdot(\vec{c}-\vec{a}) =$$

$$(2\langle 1,1,-2\rangle + 3\langle 3,-2,1\rangle)\cdot(\langle 0,1,-5\rangle - \langle 1,1,-2\rangle) =$$

$$(\langle 2,2,-4\rangle + \langle 9,-6,3\rangle)\cdot(\langle -1,0,-3\rangle) =$$

$$\langle 11,-4,-1\rangle\cdot\langle -1,0,-3\rangle =$$

$$-11 + 0 + 3 = -8$$

8. A triangle is acute if all its internal angles are less than $\frac{\pi}{2}$ and obtuse if one of its internal angles is greater than $\pi/2$. Thus, we can know whether a triangle is acute, obtuse or right if we know two of its internal angles.

If $A(2,-7,3)$, $B(-1,5,8)$, $C(4,6,-1)$ are the vertices of a triangle, then the vectors $\vec{AB}$ and $\vec{AC}$ represent two legs of the triangle ABC.

$$\vec{AB} = \langle -1-2, 5-(-7), 8-3 \rangle = \langle -3, 12, 5 \rangle$$

$$\vec{AC} = \langle 4-2, 6-(-7), -1-3 \rangle = \langle 2, 13, -4 \rangle$$

$$\vec{AB} \cdot \vec{AC} = -3(2) + 12(13) + 5(-4)$$

$$= -6 + 156 - 20 = 130$$

$$\|\vec{AB}\| = \sqrt{(-3)^2 + 12^2 + 9^2} = \sqrt{178}$$

$$\|\vec{AC}\| = \sqrt{2^2 + 13^2 + (-4)^2} = \sqrt{189}$$

$$\Rightarrow \cos\theta = \frac{\vec{AB} \cdot \vec{AC}}{\|\vec{AB}\| \, \|\vec{AC}\|} = \frac{130}{\sqrt{178}\sqrt{189}} = 0.70876$$

$\Rightarrow \theta$ = angle between $\vec{AB}$ and $\vec{AC}$ $= 0.783$ radians $< \frac{\pi}{2}$

The angle between $\vec{CA} = -\vec{AC}$ and $\vec{CB}$ is

$\vec{CB} = \langle -1-4, 5-6, 8-(-1) \rangle = \langle -5, -1, 9 \rangle$

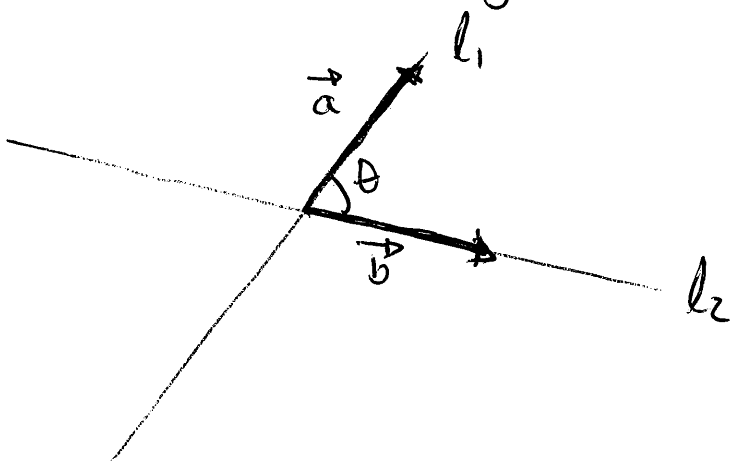
$\|\vec{CB}\| = \sqrt{(-5)^2 + (-1)^2 + (9)^2} = \sqrt{25+1+81} = \sqrt{107}$

$$\cos\gamma = \frac{-\vec{AC} \cdot \vec{CB}}{\sqrt{189}\sqrt{107}} = \frac{\langle -2, -13, 4 \rangle \cdot \langle -5, -1, 9 \rangle}{142.207} = \frac{59}{142.207}$$

$\cos\gamma = 0.4148 \Rightarrow \gamma = 1.143$ radians $< \frac{\pi}{2}$

$\therefore$ The triangle ABC is an acute triangle.

9. Since we can find vectors parallel to the lines, the angle between the lines is equal to the angle between the vectors.



l_1: $2x - y = 3 \Rightarrow y = 2x - 3 \Rightarrow y' = 2$

l_2: $3x + y = 7 \Rightarrow y = -3x + 7 \Rightarrow y' = -3$

$\vec{a} = \langle 1, 2 \rangle$

$\vec{b} = \langle 1, -3 \rangle$

Since

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{\langle 1,2\rangle \cdot \langle 1,-3\rangle}{\sqrt{1^2+2^2}\sqrt{1^2+(-3)^2}} = \frac{1-6}{\sqrt{5}\sqrt{10}}$$

$$= -0.7071 \Rightarrow \theta = 2.36 \text{ rad}$$

Since we want the acute angle, we look for the angle

$$\pi - 2.36 \text{ rad} = \underline{0.78 \text{ rad}}$$

or

$$\underline{45°}$$

10. The graphs of $\sin x$ and $\cos x$ intersect once in the interval $[0, \frac{\pi}{2}]$. The intersection point is:

$$\sin^2 x + \cos^2 x = 1, \quad \sin x = \cos x$$

$$2\sin^2 x = 1 \Rightarrow$$

$$x = \arcsin\left(\frac{1}{\sqrt{2}}\right) = 0.7853$$

intersection point: $(0.7853, 0.7071)$

Now, tangent lines to $\sin x$ and $\cos x$ at $(0.7853, 0.7071)$ are

$l_{\sin}$:

$y - y_1 = m(x - x_1)$

$y - 0.7071 = \cos(0.7853)(x - 0.7853)$

$y = 0.7071x - (0.7071)(0.7853) + 0.7071$

$y' = 0.7071$ ← slope

$\vec{a} = \langle 1, 0.7071 \rangle$ ← vector tangent to $\sin x$ at $x = 0.7853$

$l_{\cos}$:

$y - y_1 = m(x - x_1)$

$y - 0.7071 = -\sin(0.7853)(x - 0.7853)$

$y = -0.7071x + \sin(0.7853)(0.7853) + 0.7071$

$y' = -0.7071$ ← slope

$\vec{b} = \langle 1, -0.7071 \rangle$ ← vector tangent to $\cos x$ at $x = 0.7853$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} = \frac{1 - 0.5}{(1.2247)(1.2247)} = \frac{0.5}{1.5} = \frac{1}{3}$$

$\Rightarrow \theta = \arccos\left(\frac{1}{3}\right) = 1.23$ rad or 70.52 degrees.