Homework #10

Math 243 - Section 51

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1. Since the region of integration is symmetrical around the z-axis, using cylindrical coordinates seems more appropriate than using Cartesian or spherical coordinates.

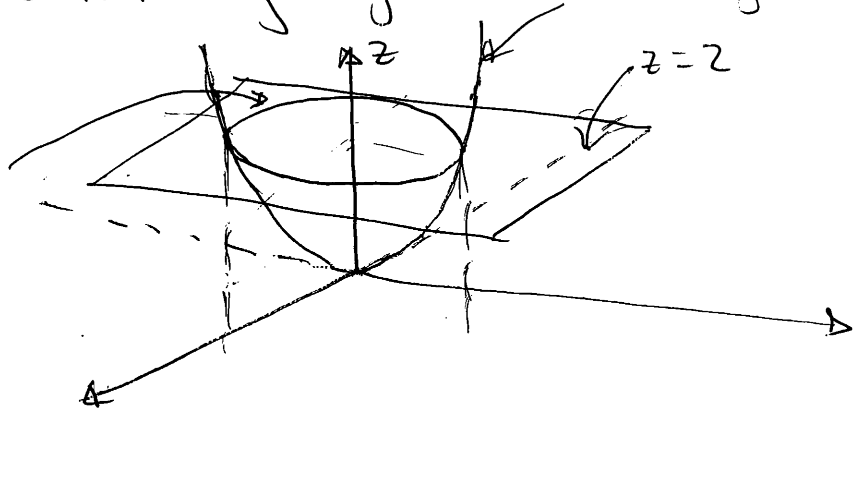
Therefore:

$$\iiint_E z\,dV = \int_0^{2\pi}\int_0^1\int_{2r^2}^{2} zr\,dz\,dr\,d\theta$$

based on the following figure:

$z = 2x^2+2y^2 = 2r^2$

$z=2$



$2 = 2x^2+2y^2 \Rightarrow$

$1 = x^2+y^2 \Rightarrow$

$1 = r^2 \Rightarrow$

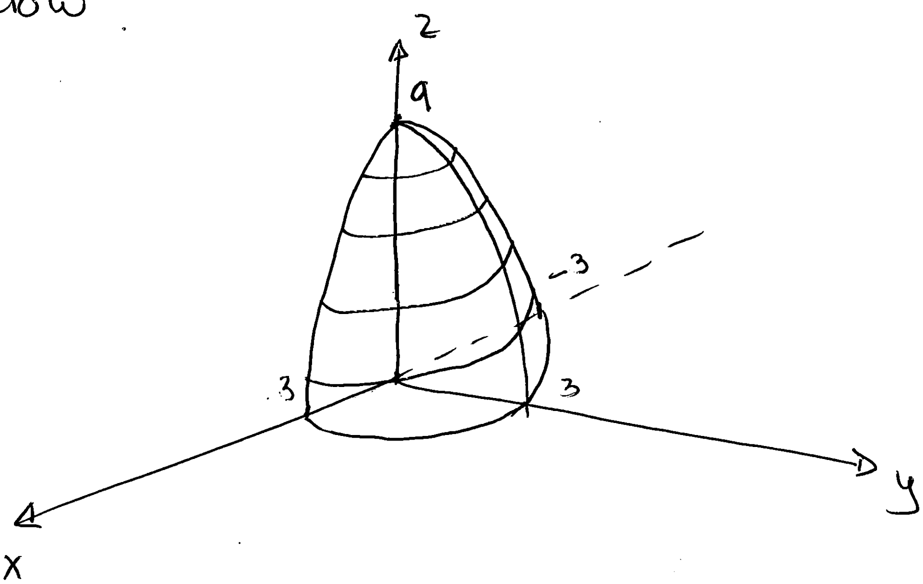
$r=1$

$$\int_0^{2\pi}\int_0^1\int_{2r^2}^{2} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{z^2}{2}r\Big]_{2r^2}^{2}\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 (2r - 2r^5)\,dr\,d\theta = \int_0^{2\pi} r^2 - \frac{r^6}{3}\Big]_0^1\,d\theta =$$

$$\int_0^{2\pi}\left(1-\frac{1}{3}\right)d\theta = \int_0^{2\pi}\frac{2}{3}\,d\theta = \frac{2}{3}\,\theta\Big|_0^{2\pi} = \frac{4\pi}{3}$$

2. The region of integration in this problem is depicted below:



Since this region is symmetric around the z-axis, we can use cylindrical coordinates.

Using the transformations

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

we obtain:

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx =$$

$$\int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2} r\, r\, dz\, dr\, d\theta = \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2} r^2\, dz\, dr\, d\theta =$$

$$\int_{0}^{\pi}\int_{0}^{3} r^2 z\Big|_{0}^{9-r^2} dr\, d\theta = \int_{0}^{\pi}\int_{0}^{3} r^2(9-r^2)\, dr\, d\theta =$$

$$\int_{0}^{\pi}\int_{0}^{3} (9r^2 - r^4)\, dr\, d\theta = \int_{0}^{\pi} 3r^3 - \frac{r^5}{5}\Big|_{0}^{3} d\theta =$$

$$\int_{0}^{\pi}\left(81 - \frac{243}{5}\right) d\theta = \frac{162}{5}\,\theta\Big|_{0}^{\pi} = \frac{162\pi}{5}$$

3. Using spherical coordinates:

The integrand becomes

$$\rho e^{-\rho^2}$$

so

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)}\, dx\, dy\, dz =$$

$$\lim_{a\to\infty} \int_0^{2\pi}\int_0^{a}\int_0^{\pi} \rho e^{-\rho^2} \rho^2 \sin\phi \, d\phi\, d\rho\, d\theta =$$

$$\lim_{a\to\infty} \int_0^{2\pi}\int_0^{a} \rho^3 e^{-\rho^2} (-\cos\phi)\Big|_0^{\pi} d\rho\, d\theta =$$

$$\lim_{a\to\infty} \int_0^{2\pi}\int_0^{a} 2\rho^3 e^{-\rho^2} d\rho\, d\theta;$$ Integrating $2\rho^3 e^{-\rho^2}$ by parts with

$u=\rho^2 \quad u'=2\rho$

$v'=2\rho e^{-\rho^2} \quad v=-e^{-\rho^2}$

We obtain:

$$\lim_{a\to\infty} \int_0^{2\pi} -e^{-\rho^2}(\rho+1)\Big|_0^{a} d\theta =$$

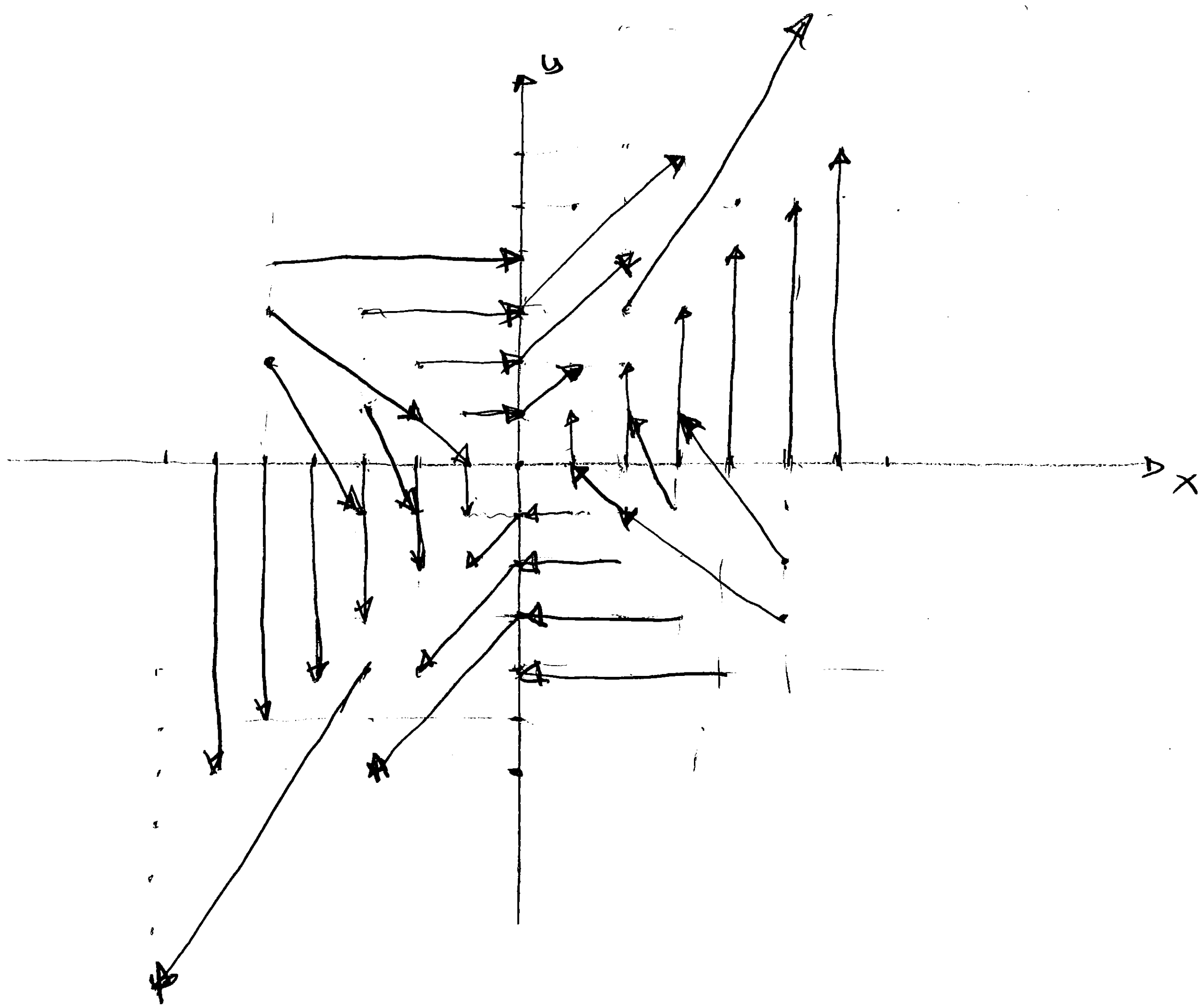
$$\lim_{a\to\infty} \int_0^{2\pi} \left[-e^{-a^2}(a+1) + e^0(0+1)\right] d\theta =$$

$$\lim_{a\to\infty} \int_0^{2\pi} \left[-e^{-a^2}(a+1) + 1\right] d\theta =$$

$$\lim_{a\to\infty} -\theta e^{-a^2}(a+1) + \theta \Big]_0^{2\pi} =$$

$$\lim_{a\to\infty} -2\pi e^{-a^2}(a+1) + 2\pi = \underline{2\pi} \text{ because } \lim_{a\to\infty} e^{-a^2} = 0$$

4.



5.

$$\int_C xyz\, ds = \int_0^{\pi} (2\sin t)(t)(-2\cos t)\sqrt{5}\, dt$$

$$\vec{r}(t) = \langle 2\sin t, t, -2\cos t\rangle$$

$$\vec{r}\,'(t) = \langle 2\cos t, 1, 2\sin t\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{4\cos^2 t + 1 + 4\sin^2 t}$$

$$= \sqrt{4+1} = \sqrt{5}$$

since $\sin(2t) = 2\sin t\cos t$:

$$\sqrt{5}\int_0^{\pi} -4t\sin t\cos t\, dt = \sqrt{5}\int_0^{\pi} -2t\sin(2t)\, dt$$

By parts: $u = t \quad u' = 1$

$v' = -2\sin(2t) \quad v = \cos(2t)$

$$\therefore \sqrt{5}\int_0^{\pi} -2t\sin(2t)\,dt = \left(t\cos(2t)\Big|_0^{\pi} - \int_0^{\pi}\cos(2t)\,dt\right)\sqrt{5}$$

$$= \left((\pi\cos(2\pi) - 0) - \frac{1}{2}\int_0^{2\pi}\cos u\, du\right)\sqrt{5}$$

$$= \sqrt{5}\left(\pi - \frac{1}{2}(-\sin u)\Big|_0^{2\pi}\right) = \left(\pi - \frac{1}{2}(0-0)\right)\sqrt{5} = \underline{\sqrt{5}\,\pi}$$

6. $\int_C y\,dx + z\,dy + x\,dz$, $x=\sqrt{t}$, $y=t$, $z=t^2$

$0 \le t \le 4$

$dx = \frac{1}{2\sqrt{t}}\,dt$; $dy = dt$; $dz = 2t\,dt$

$\therefore$ $\int_C y\,dx + z\,dy + x\,dz = \int_0^4 t\frac{1}{2\sqrt{t}}dt + t^2\,dt + \sqrt{t}\,(2t)\,dt =$

$\int_0^4 \left(\frac{1}{2}\sqrt{t} + t^2 + 2t^{3/2}\right)dt = \frac{1}{3}t^{3/2} + \frac{t^3}{3} + \frac{4}{5}t^{5/2}\Big]_0^4 = \frac{248}{5}$

7. $\vec{r}(t) = \langle t^2 , t^3 \rangle$

$\vec{r}\,'(t) = \langle 2t, 3t^2 \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle e^{t^2-1}, t^5 \rangle \cdot \langle 2t, 3t^2 \rangle =$

$2te^{t^2-1} + 3t^7$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (2te^{t^2-1} + 3t^7)\,dt = e^{t^2-1} + \frac{3}{8}t^8\Big]_0^1 = \boxed{\frac{11}{8} - \frac{1}{e}}$

8. $\int_C \vec{F} \cdot d\vec{r} = W \qquad C: \vec{r}(t) = \langle t^2+1,\ t \rangle$

$$\vec{r}\,'(t) = \langle 2t,\ 1 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle (t^2+1)^2,\ t e^{t^2+1} \rangle$$

$$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 2t(t^2+1)^2 + t e^{t^2+1}$$

$$\therefore W = \int_0^1 \left(2t(t^2+1)^2 + t e^{t^2+1}\right) dt = \frac{t^6}{3} + t^4 + t^2 + \frac{1}{2}\left(e^{t^2+1}\right)\Big|_0^1$$

$$= \underline{\frac{7}{3} + \frac{1}{2}(e-1)e}$$

9. $C: \vec{r}(t) = \langle 2\cos t,\ 2\sin t \rangle$

$$\vec{r}\,'(t) = \langle -2\sin t,\ 2\cos t \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 4\cos^2 t,\ 4\sin t \cos t \rangle$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -8\cos^2 t \sin t + 8\cos^2 t \sin t = 0$$

$$\therefore W = \int_C \vec{F} \cdot d\vec{r} = 0$$

10. $\vec{r}(t) = \langle \cos t, \sin t \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -a \sin t + b \cos t$

$$W = \int_0^{2\pi} (-a \sin t + b \cos t)\, dt = a \cos t \Big|_0^{2\pi} + b \sin t \Big|_0^{2\pi}$$

$$= a(1-1) + b(0-0)$$

$$= \underline{0}$$