Homework #11

Math 243 - Section 51

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1. $\vec{F}$ is conservative if $\nabla \times \vec{F} = \vec{0}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + y^{-2} & x^2 - 2xy^{-3} & 0 \end{vmatrix} =$$

$$\hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left(2x - 2y^{-3} - (2x - 2y^{-3})\right) =$$

$$\vec{0}$$

$$f_x = 2xy - y^{-2} \Rightarrow f_c = \int (2xy - y^{-2})\,dx = x^2y - xy^{-2} + C(y)$$

$$\frac{\partial f_c}{\partial y} = x^2 + 2xy^{-3} + C'(y) = f_y \Rightarrow C'(y) = 0 \Rightarrow C(y) = K$$

$$\therefore\ f(x,y) = x^2y - xy^{-2} + K$$

2. Since $Q_x - P_y = 0$, $\vec{F}$ is conservative and there is an f such that $\nabla f = \vec{F}$.

So, if $f_x = x^2 \Rightarrow f_c = \int x^2 dx = \frac{x^3}{3} + C(y) \Rightarrow$

$\frac{\partial f_c}{\partial y} = C'(y) = f_y = y^2 \Rightarrow C(y) = \frac{y^3}{3} + K.$

So, $f = \frac{x^3}{3} + \frac{y^3}{3} + K$

$\therefore$ By the FTC4LIs:

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(2,8) - f(-1,2) = \frac{8}{3} + \frac{512}{3} - \left(\frac{-1}{3} + \frac{8}{3}\right) =$

$\frac{513}{3} = 171$

3. Showing that $\int_C \vec{F} \cdot d\vec{r}$ is path independent means showing that $\nabla \times \vec{F} = \vec{0}$.

In this case, since it is a two-dimensional field, it is enough to show $Q_x - P_y = 0$

So $Q = \frac{1}{2}(x^2e^y + y^2)$ and $P = xe^y$.

$Q_x = xe^y$ and $P_y = xe^y \Rightarrow Q_x - P_y = 0$

$\therefore \int_C xe^y\,dx + \left(\frac{1}{2}(x^2e^y + y^2)\right)dy$ is path-independent.

4. By the same token, if $\nabla \times \vec{F} \neq \vec{0}$ then
$\int_C \vec{F} \cdot d\vec{r}$ is not path-independent.

In this case, $Q = \frac{1}{2}(x^2e^y + xy^2)$ and $P = x^2e^y$

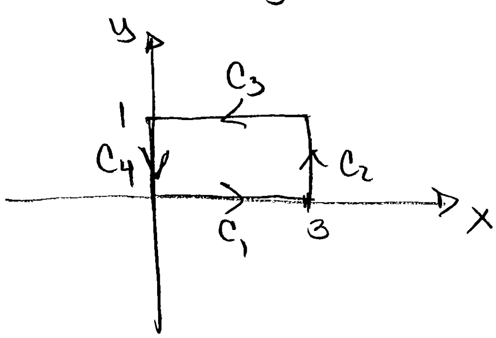
$Q_x = xe^y + \frac{1}{2}y^2$; $P_y = x^2e^y \Rightarrow Q_x - P_y \neq 0 \therefore$

$\int_C x^2e^y\,dx + \left(\frac{1}{2}(x^2e^y + xy^2)\right)dy$ is path-dependent.

5. Let $\vec{F} = \langle f(x), g(y), h(z) \rangle$, then

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x) & g(y) & h(z) \end{vmatrix} = \left(\frac{\partial}{\partial y} h(z) - \frac{\partial}{\partial z} g(y)\right)\hat{i} - \left(\frac{\partial}{\partial x} h(z) - \frac{\partial}{\partial z} f(x)\right)\hat{j} + \left(\frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x)\right)\hat{k} = \underline{\vec{0}}$$

6. Directly:



$$\oint_C xy\,dx + x^3\,dy = \int_{C_1} xy\,dx + x^3\,dy + \int_{C_2} xy\,dx + x^3\,dy + \int_{C_3} xy\,dx + x^3\,dy + \int_{C_4} xy\,dx + x^3\,dy$$

Along C_1: $dy = 0$

Along C_2: $dx = 0$

Along C_3: $dy = 0$

Along C_4: $dx = 0$

$$\therefore \oint_C xy\,dx + x^3\,dy = \int_{C_1} xy\,dx + \int_{C_2} x^3\,dy + \int_{C_3} xy\,dx + \int_{C_4} x^3\,dy$$

$\int_{C_1} xy\,dx = 0$ because $y=0$ along C_1

$$\int_{C_2} x^3\,dy = \int_0^1 (3)^3\,dy = 27y\Big|_0^1 = 27$$

$$\int_{C_3} xy\,dx = \int_3^0 x(1)\,dx = \frac{x^2}{2}\Big|_3^0 = -\frac{9}{2}$$

$\int_{C_4} x^3\,dy = 0$ because $x=0$ along C_1

$\therefore$

$$\oint_C xy\,dx + x^3\,dy = 27 - \frac{9}{2} = \frac{54}{2} - \frac{9}{2} = \boxed{\frac{45}{2}}$$

By Green's theorem:

$$\oint_C xy\,dx + x^3\,dy = \iint_D \left(\frac{\partial}{\partial x}x^3 - \frac{\partial}{\partial y}xy\right) dA,$$ where

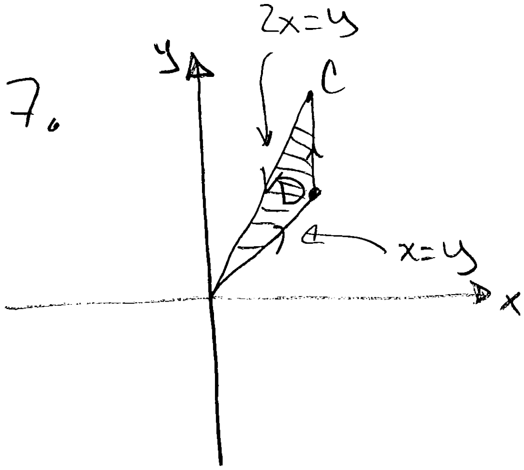
D is the rectangle bouded by C.

Thus

$$\iint_D \left(\frac{\partial}{\partial x}x^3 - \frac{\partial}{\partial y}xy\right) dA = \int_0^3\int_0^1 (3x^2 - x)\,dy\,dx =$$

$$\int_0^3 (3x^2 - x)\, dx = x^3 - \frac{x^2}{2}\Big|_0^3 = 27 - \frac{9}{2} = \underline{\frac{45}{2}}$$

7.



$$\oint_C xy^2\, dx + 8x^2 y\, dy = \iint_D (16xy - 2xy)\, dA =$$

$$\iint_D 14xy\, dA = \int_0^2 \int_x^{2x} 14xy\, dy\, dx = \int_0^2 7xy^2\Big|_x^{2x} dx =$$

$$\int_0^2 7x(4x^2 - x^2)\, dx = 21\int_0^2 x^3\, dx = \frac{21}{4} x^4\Big|_0^2 = 4(21) = \underline{84}$$

8.

$$\oint_C y^4 dx + 2xy^3 dy = \iint_D (2y^3 - 4y^3)\, dA = \iint_D -2y^3\, dA$$

Now, $x^2 + 2y^2 = 2 \Rightarrow \frac{x^2}{2} + y^2 = 1 \Rightarrow \left(\frac{x}{\sqrt{2}}\right)^2 + y^2 = 1$

if $u = \frac{x}{\sqrt{2}}$, we obtain: $u^2 + y^2 = 1$ ∴

$$\iint_D -2y^3\, dA = \sqrt{2}\int_0^{2\pi}\int_0^1 -2(r\sin\theta)^3\, r\, dr\, \theta =$$

$$-2\sqrt{2}\int_0^{2\pi}\int_0^1 r^4 \sin^3\theta\, dr\, d\theta = -2\sqrt{2}\int_0^{2\pi} \frac{1}{5}\sin^3\theta\, d\theta =$$

$$-\frac{2}{5}\sqrt{2}\int_0^{2\pi} \sin^3\theta\, d\theta = -\frac{2}{5}\sqrt{2}\int_0^{2\pi} \sin^2\theta \sin\theta\, d\theta = -\frac{2\sqrt{2}}{5}\int_0^{2\pi}(1-\cos^2\theta)\sin\theta\, d\theta$$

$$-\frac{2\sqrt{2}}{5}\left[\int_0^{2\pi}\sin\theta\, d\theta - \int_0^{2\pi}\cos^2\theta\sin\theta\, d\theta\right] = -\frac{2\sqrt{2}}{5}\left[-\cos\theta\Big|_0^{2\pi} - \frac{\cos^3\theta}{3}\Big|_0^{2\pi}\right]$$

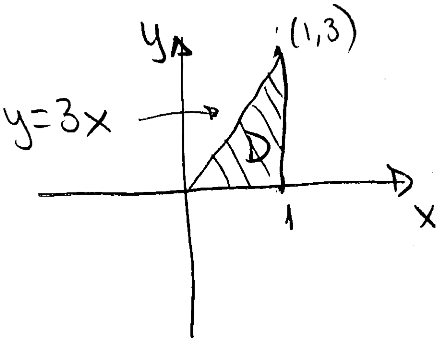
$$= \underline{0}$$

9. Since C is closed, we can use Green's theorem. Thus,

$$\oint_C (e^x + 2y)\,dx + (8x - \tan y)\,dy = \iint_D (8-2)\,dA = 6\iint_D dA =$$

$$6(5) = \underline{30}$$

10. $$\oint_C \sqrt[5]{1+x^7}\,dx + xy^2\,dy = \iint_D (y^2 - 0)\,dA = \iint_D y^2\,dA$$



$$\Rightarrow \iint_D y^2\,dA = \int_0^1 \int_0^{3x} y^2\,dy\,dx =$$

$$\int_0^1 \frac{y^3}{3}\Big|_0^{3x} dx = \int_0^1 \frac{27x^3}{3}\,dx = 9\,\frac{x^4}{4}\Big|_0^1 = \boxed{\frac{9}{4}}$$