Homework #12
Math 243 - Section 51
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1. $\vec{r}(x,y) = \langle x, y, 5-2x-y \rangle$

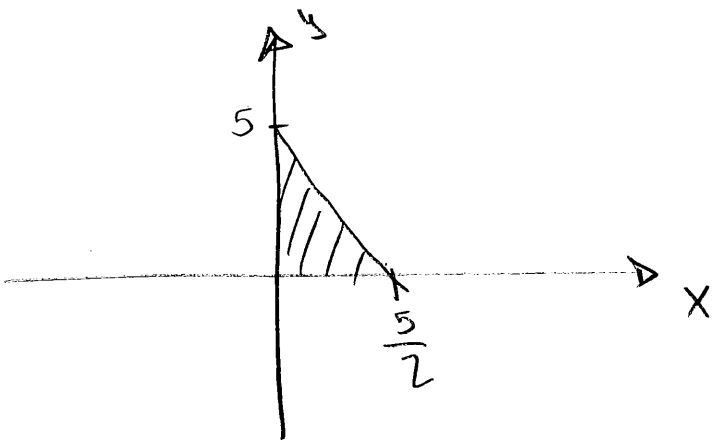
$\vec{r}_x(x,y) = \langle 1, 0, -2 \rangle$; $\vec{r}_y(x,y) = \langle 0, 1, -1 \rangle$

$$\vec{n} = \vec{r}_x(x,y) \times \vec{r}_y(x,y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = \langle 2, 1, 1 \rangle$$

$$\|\vec{n}\| = \sqrt{4+1+1} = \sqrt{6}$$

Surface Area: $SA = \iint_S 1 \, dS = \iint_D \sqrt{6} \, dA$

D:



$$SA = \int_0^{\frac{5}{2}}\int_0^{5-2x} \sqrt{6}\, dy\, dx = \sqrt{6}\int_0^{\frac{5}{2}} y\Big|_0^{5-2x} dx = \sqrt{6}\int_0^{\frac{5}{2}} (5-2x)\, dx =$$

$$\sqrt{6}\left(5x - x^2\right)\Big|_0^{\frac{5}{2}} = \sqrt{6}\left(\frac{25}{2} - \frac{25}{4}\right) = \boxed{\frac{25\sqrt{6}}{4}}$$

2. $\vec{r}(x,y) = \langle x, y, x+y^2 \rangle$

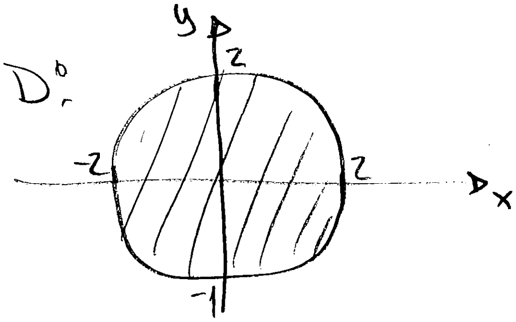
$\vec{r}_x(x,y) = \langle 1, 0, 1 \rangle$; $\vec{r}_y(x,y) = \langle 0, 1, 2y \rangle$

$$\vec{n} = \vec{r}_x(x,y) \times \vec{r}_y(x,y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2y \end{vmatrix} = \langle -1, -2y, 1 \rangle$$

$$\|\vec{n}\| = \sqrt{1 + 4y^2 + 1} = \sqrt{2 + 4y^2}$$

$$SA = \iint_S 1\, dS = \iint_D \sqrt{2+4y^2}\, dA$$

D:



Using polar coordinates:

$$\iint_D \sqrt{2+4y^2}\, dA = \int_0^{2\pi}\int_0^2 \sqrt{2+4r^2\sin^2\theta}\; r\, dr\, d\theta$$

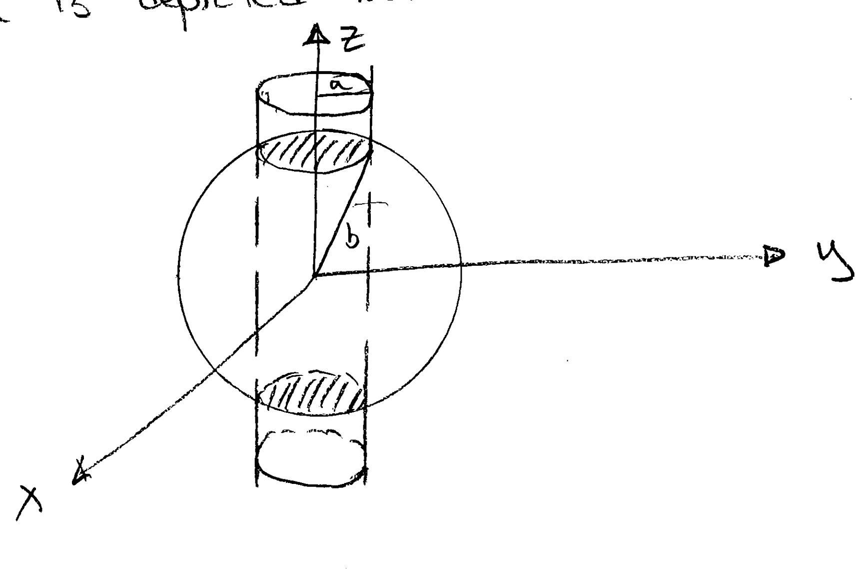
or

$$\iint_D \sqrt{2+4y^2}\, dA = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{2+4y^2}\; dx\, dy$$

In either case, the integral cannot be expressed in terms of elementary functions. However, it can be solved numerically giving us:

$$\iint_D \sqrt{2+4y^2}\, dA \simeq \underline{29.203}$$

3. The situation is depicted below



$SA = \iint_S dS$, where S is the part of the sphere of radius b inside a cylinder of radius a.

A parametric representation of the sphere is

$$\vec{r}(\phi,\theta) = \langle b\sin\phi\cos\theta,\ b\sin\phi\sin\theta,\ b\cos\phi \rangle$$

so

$$\vec{r}_\phi(\phi,\theta) = \langle b\cos\phi\cos\theta,\ b\cos\phi\sin\theta,\ -b\sin\phi \rangle$$

$$\vec{r}_\theta(\phi,\theta) = \langle -b\sin\phi\sin\theta,\ b\sin\phi\cos\theta,\ 0 \rangle$$

$$\vec{n} = \vec{r}_\phi(\phi,\theta) \times \vec{r}_\theta(\phi,\theta) = \langle b^2\sin^2\phi\cos\theta,\ b^2\sin^2\phi\sin\theta,\ b^2\sin\phi\cos\phi\cos^2\theta + b^2\sin\phi\cos\phi\sin^2\theta \rangle$$

$$= \langle b^2\sin^2\phi\cos\theta,\ b^2\sin^2\phi\sin\theta,\ b^2\sin\phi\cos\phi \rangle$$

$$\|\vec{n}\| = \sqrt{b^4\sin^4\phi\cos^2\theta + b^4\sin^4\phi\sin^2\theta + b^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{b^4\sin^4\phi + b^4\sin^2\phi\cos^2\phi} = \sqrt{b^4\sin^2\phi\sin^2\phi + b^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{b^4\sin^2\phi} = b^2\sin\phi$$

$\therefore$

$$SA = \iint_S dS = \iint_D b^2\sin\phi \, dA$$

From the figure, you shold notice that ϕ depens on a. Thus, we need to find a function f such that

$\phi = f(a)$ so that

$$SA = \int_0^{2\pi}\int_0^{f(a)} b^2 \sin\phi \, d\phi \, d\theta$$

We know:

$$x^2+y^2=a^2$$
$$x^2+y^2+z^2=b^2$$

so

$$a^2+z^2=b^2$$

but $z = b\cos\phi$, thus

$$a^2+b^2\cos^2\phi = b^2$$
$$\cos^2\phi = 1-\frac{a^2}{b^2} = 1-\left(\frac{a}{b}\right)^2$$
$$\cos\phi = \sqrt{1-\left(\frac{a}{b}\right)^2}$$

By taking the positive part we are restricting ourselves to the range $0 \leq \phi \leq \frac{\pi}{2}$. Thus, we will need to multiply by 2 at the end.

$$\phi = \arccos\left(\sqrt{1-\left(\frac{a}{b}\right)^2}\right)$$

So

$$\frac{SA}{2} = \int_0^{2\pi}\int_0^{\arccos\left(\sqrt{1-\left(\frac{a}{b}\right)^2}\right)} b^2\sin\phi \, d\phi \, d\theta$$

$$\frac{SA}{2} = -\int_0^{2\pi} b^2 \cos\phi \Big|_0^{\arccos\left(\sqrt{1-\left(\frac{a}{b}\right)^2}\right)} d\theta = -\int_0^{2\pi} b^2 \cos\left(\arccos\left(\sqrt{1-\left(\frac{a}{b}\right)^2}\right)\right) - b^2\cos(0)\ d\theta =$$

$$= -\int_0^{2\pi}\left(b^2\sqrt{1-\left(\frac{a}{b}\right)^2} - b^2\right) d\theta = b^2\left(\sqrt{1-\left(\frac{a}{b}\right)^2} - 1\right)\left(-2\pi\right) =$$

$$2\pi b^2\left(1 - \sqrt{1-\left(\frac{a}{b}\right)^2}\right)$$

$$\therefore SA = \underline{4\pi b^2\left(1 - \sqrt{1-\left(\frac{a}{b}\right)^2}\right)}$$

4. $\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ u\rangle$, $0 \le u \le 1$, $0 \le v \le \frac{\pi}{2}$

$\vec{r}_u(u,v) = \langle \cos v, \sin v, 1\rangle$

$\vec{r}_v(u,v) = \langle -u\sin v, u\cos v, 0\rangle$

$$\vec{n} = \vec{r}_u(u,v) \times \vec{r}_v(u,v) = \langle -u\cos v, -u\sin v, u\cos^2 v + u\sin^2 v\rangle =$$

$$= \langle -u\cos v, -u\sin v, u\rangle$$

$$\|\vec{n}\| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{u^2+u^2} = u\sqrt{2}$$

$$xyz = (u\cos v)(u\sin v)(u) = u^3 \sin v \cos v$$

$\therefore$

$$\iint_S xyz\, dS = \iint_D u^3 \sin v \cos v \left(u\sqrt{2}\right) dA =$$

$$\sqrt{2}\int_0^{\pi/2}\int_0^1 u^4 \sin v \cos v \, du\, dv = \sqrt{2}\int_0^{\pi/2} \frac{1}{5} \sin v \cos v \, dv =$$

$$\frac{\sqrt{2}}{5}\int_0^{\pi/2} \sin v \cos v \, dv \overset{\left\{\substack{w = \sin v \\ dw = \cos v\, dv}\right\}}{=} \frac{\sqrt{2}}{5}\int_0^1 w\, dw = \boxed{\frac{\sqrt{2}}{10}}$$

5. S is a closed surface. Therefore, we can apply the Divergence Theorem:

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_E \nabla\cdot\vec{F}\, dV$$

So $\nabla\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle x, 2y, 3z\rangle = 1+2+3 = 6$

So $$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_E 6\, dV = 6\overbrace{\iiint_E dV}^{\text{Volume of cube}} = 6(8) = \underline{48}$$

6. $S: x^2+y^2+z^2=4$, $z>0$, whose parametric representation is

$$\vec{r}(\phi,\theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle$$

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi/2$$

From previous exercises we know

$$\vec{n} = \vec{r}_\phi(\phi,\theta) \times \vec{r}_\theta(\phi,\theta) = \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi \rangle$$

Now,

$$\vec{v}(\vec{r}(\phi,\theta)) = \langle 2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 4 \rangle$$

$$\text{flux} = \iint_S \rho\vec{v}\cdot d\vec{S} = \iint_S 1200\vec{v}\cdot d\vec{S} = 1200\iint_D \vec{v}\cdot\vec{n}\, dA =$$

$$1200\int_0^{\pi/2}\int_0^{2\pi} \langle 2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 4\rangle\cdot\langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle \, d\theta\, d\phi$$

$$= 1200\int_0^{\pi/2}\int_0^{2\pi} (16\sin^3\phi\sin\theta\cos\theta + 16\sin\phi\cos\phi)\, d\theta\, d\phi =$$

$$19200\int_0^{\pi/2}\left[\cancel{\left[\frac{\sin^2\theta}{2}\right]_0^{2\pi}\sin^3\phi}^{\,0} + 2\pi\sin\phi\cos\phi\right] d\phi =$$

$$= 38400\pi \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} = \underline{19200\pi \ \text{Kg}/\text{s}}$$

7. $\nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle x^2yz, xy^2z, xyz^2 \right\rangle =$

$$2xyz + 2xyz + 2xyz = 6xyz$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F} \, dV = \iiint_E 6xyz \, dV =$$

$$\int_0^1 \int_0^1 \int_0^1 6xyz \, dx \, dy \, dz = \int_0^1 \int_0^1 3x^2yz \Big|_0^1 dy \, dz =$$

$$\int_0^1 \int_0^1 3yz \, dy \, dz = \int_0^1 \frac{3}{2} y^2 z \Big|_0^1 dz = \int_0^1 \frac{3}{2} z \, dz =$$

$$\frac{3}{4} z^2 \Big|_0^1 = \boxed{\frac{3}{4}}$$

8. $\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle x^3+y^3, y^3+z^3, z^3+x^3 \right\rangle =$

$$3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2)$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F} \, dV = 3\iiint_E (x^2+y^2+z^2) \, dV$$

using spherical coordinates:

$$3\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho^2(\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta = 3\int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5}\Big|_0^2 \sin\phi \, d\phi \, d\theta =$$

$$\frac{3(2)^5}{5}\int_0^{2\pi} -\cos\phi \Big|_0^{\pi} \, d\theta = \frac{3(2)^6}{5}\int_0^{2\pi} d\theta = \frac{3(2)^7\pi}{5} = \underline{\frac{384}{5}\pi}$$

9. $S: \vec{r}(\phi,\theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle$

From previous exercises:

$$\|\vec{n}\| = (2)^2 \sin\phi = 4\sin\phi$$

$$\therefore \iint_S (2x+2y+z^2) \, dS = \iint_D \left(2(2\sin\phi\cos\theta) + 2(2\sin\phi\sin\theta) + (2\cos\phi)^2\right) \cdot (4\sin\phi) \, dA$$

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$$= \iint_D \left[16\sin^2\phi(\cos\theta + \sin\theta) + 16\cos^2\phi\sin\phi \; dA\right.$$

$$\int_0^{\pi}\int_0^{2\pi} \left[16\sin^2\phi\,(\cos\theta + \sin\theta) + 16\cos^2\phi\sin\phi\right] d\theta\, d\phi$$

$$\int_0^{\pi} 16\sin^2\phi \left(\cancelto{0}{\int_0^{2\pi} (\cos\theta + \sin\theta)\, d\theta} \right) + \int_0^{\pi} 32\pi\cos^2\phi\sin\phi\, d\phi =$$

$$32\pi \int_0^{\pi} \cos^2\phi \sin\phi\, d\phi \; ; \quad \begin{aligned} u &= \cos\phi \\ du &= -\sin\phi\, d\phi \end{aligned} \quad \Rightarrow$$

$$-32\pi \int_1^{-1} u^2\, du = -32\pi \left(\frac{u^3}{3}\right)\Bigg|_1^{-1} = -32\pi\left(-\frac{1}{3} - \frac{1}{3}\right) = \boxed{\frac{64\pi}{3}}$$

10. Since is the boundary of a solid region, S is closed. So, by the Divergence Theorem:

$$\oiint_S \nabla \times \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot (\nabla \times \vec{F})\, dV$$

Now, assume $\vec{F} = \langle P, Q, R \rangle$ where P, Q, and R are functions of x, y, and z with continuous 2nd order partial derivatives.

Then

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\hat{i} - (R_x - P_z)\hat{j} + (Q_x - P_y)\hat{k} =$$

$$\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\nabla \cdot \nabla \times \vec{F} = \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(P_z - R_x) + \frac{\partial}{\partial z}(Q_x - P_y)$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

Since these derivatives are continuous:

$$R_{xy} = R_{yx} \quad Q_{zx} = Q_{xz} \quad P_{zy} = P_{yz}$$

So

$$\nabla \cdot \nabla \times \vec{F} = 0$$

$$\therefore \oiint_S \nabla \times \vec{F} \cdot d\vec{S} = 0$$