Homework #3

Math 243 - Section 51

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1. $2(x-5)+1(y-3)-1(z-5)=0$

$2x-10+y-3-z+5=0$

$2x+y-z-8=0$

2. We can use the line's direction vector as the plane's normal vector. Thus, given that

$$L:\ \vec{r}(t)=\langle 3t,\ 2-t,\ 3+4t\rangle=\langle 0,2,3\rangle+t\underbrace{\langle 3,-1,4\rangle}_{\vec{n}}$$

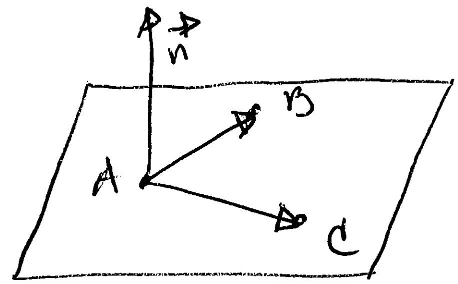
Therefore, an equation of the plane is:

$3(x-2)-1(y-0)+4(z-1)=0$

$3x-6-y+4z-4=0$

$3x-y+4z-10=0$

3. The situation is shown below



So, $\vec{n} = \vec{AC} \times \vec{AB}$.

$\vec{AC} = \langle -1-3, -2-(-1), -3-2 \rangle = \langle -4, -1, -5 \rangle$

$\vec{AB} = \langle 8-3, 2-(-1), 4-2 \rangle = \langle 5, 3, 2 \rangle$

$$\vec{n} = \vec{AC} \times \vec{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -1 & -5 \\ 5 & 3 & 2 \end{vmatrix} = (-2-(-15))\hat{i} - (-8-(-25))\hat{j} + (-12-(-5))\hat{k} = \langle 13, -17, -7 \rangle$$

An equation of the plane is:

$$13(x-3) - 17(y+1) - 7(z-2) = 0$$

$$13x - 39 - 17y - 17 - 7z + 14 = 0$$

$$13x - 17y - 7z - 36 = 0$$

4. A point on the line of intersection can be found if we set one variable to zero and solve the resulting system of equations.

We can try setting $x=0$, so the equations of the planes become

$y-z=2 \quad (1)$

$-y+3z=1 \quad (2)$

From (1), $y=2+z$

Substituting y in (2):

$-(2+z)+3z=1$

$-2-z+3z=1$

$2z=3 \Rightarrow z=\frac{3}{2}$

$\therefore \; y=2+\frac{3}{2}=\frac{4+3}{2}=\frac{7}{2}$

So, a point on the intersection line is

$\left(0, \frac{7}{2}, \frac{3}{2}\right)$

The direction vector of the line is perpendicular to both of the normal vectors.

Therefore: $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$\vec{n}_1 = \langle 1, 1, -1 \rangle$; $\vec{n}_2 = \langle 2, -1, 3 \rangle$

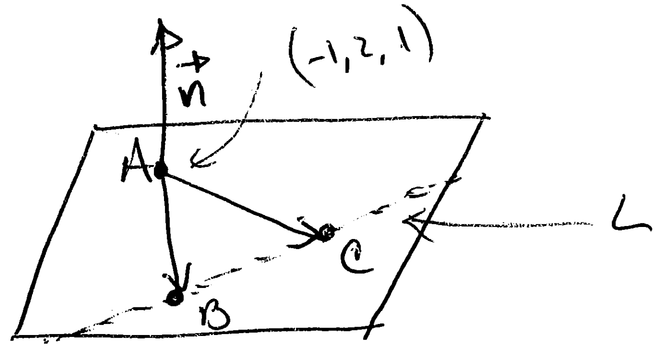
$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} = (3-1)\hat{i} - (3-(-2))\hat{j} + (-1-2)\hat{k} =$$

$\langle 2, -5, -3 \rangle$

Therefore, the line of intersection is

$L: \langle 0, \frac{7}{2}, \frac{3}{2} \rangle + t\langle 2, -5, -3 \rangle$

So, we have the following situation:



We can use the line to generate 2 points (B and C) so we can calculate $\vec{n}$.

B: $t=0$ in $L \Rightarrow$ B has coordinates $\left(0, \frac{7}{2}, \frac{3}{2}\right)$

C: $t=1$ in $L \Rightarrow$ C has coordinates $\left(2, -\frac{3}{2}, -\frac{3}{2}\right)$

Then

$$\vec{AB} = \left\langle 0-(-1), \frac{7}{2}-2, \frac{3}{2}-1 \right\rangle$$

$$= \left\langle 1, \frac{3}{2}, \frac{1}{2} \right\rangle$$

$$\vec{AC} = \left\langle 2-(-1), -\frac{3}{2}-2, -\frac{3}{2}-1 \right\rangle$$

$$= \left\langle 3, -\frac{7}{2}, -\frac{5}{2} \right\rangle$$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3/2 & 1/2 \\ 3 & -7/2 & -5/2 \end{vmatrix}$$

$$= \left(\frac{-15}{4} - \left(\frac{-7}{4}\right)\right)\hat{i} - \left(\frac{-5}{2} - \frac{3}{2}\right)\hat{j} + \left(\frac{-7}{2} - \frac{9}{2}\right)\hat{k}$$

$$= \langle -2, 4, -8 \rangle$$

Therefore, an equation of the plane is

$$-2(x+1)+4(y-2)-8(z-1)=0$$

$$-2x-2+4y-8-8z+8=0$$

$$-2x+4y-8z-2=0$$

or

$$-x+2y-4z-1=0$$

5. $\overbrace{(1+2t)}^{x}+2\overbrace{(4t)}^{y}-\overbrace{(2-3t)}^{z}+1=0$

$$1+2t+8t-2+3t+1=0$$

$$13t=0 \Rightarrow t=0$$

The point of intersection is

$x=1$

$y=0$

$z=2$

6. Let's find the equation of L_1.

$$\vec{v} = \langle 2-1, 4-2, 8-6 \rangle = \langle 1, 2, 2 \rangle$$

$$L_1: \vec{r} = \langle 1, 2, 6 \rangle + t\langle 1, 2, 2 \rangle = \langle 1+t, 2+2t, 6+2t \rangle$$

Now, let's find the equation of π_2.

$$\vec{n} = \vec{AB} \times \vec{AC}; \quad \vec{AB} = \langle 0-3, 0-2, 1-(-1) \rangle = \langle -3, -2, 2 \rangle$$

$$\vec{AC} = \langle 1-3, 2-2, 1-(-1) \rangle = \langle -2, 0, 2 \rangle$$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -2 & 2 \\ -2 & 0 & 2 \end{vmatrix} = (-4-0)\hat{i} - (-6-(-4))\hat{j} + (0-4)\hat{k} = \langle -4, 2, -4 \rangle$$

So π_2: Taking $(0,0,1)$ as point:

$$-4(x-0) + 2(y-0) - 4(z-1) = 0$$

$$-4x + 2y - 4z + 4 = 0 \quad \text{or} \quad -2x + y - 2z + 2 = 0$$

Now that we have the equations of both planes, we can find an equation for their line of intersection:

If we set $z=0$, we have

π_1: $x-y+1=0 \Rightarrow y=x+1$

π_2: $-2x+y+2=0$

y in π_2:

$$-2x+(x+1)+2=0$$

$$-2x+x+3=0$$

$$-x+3=0 \Rightarrow x=3 \text{ and therefore } y=4$$

a point on the intersection is thus $(3,4,0)$.

The direction vector of the intersection line is

$$\vec{v} = \underset{\substack{\uparrow \\ \text{normal} \\ \text{vector} \\ \text{of} \\ \pi_1}}{\vec{n}_1} \times \underset{\substack{\uparrow \\ \text{normal} \\ \text{vector} \\ \text{of} \\ \pi_2}}{\vec{n}_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ -2 & 1 & -2 \end{vmatrix} = (2-2)\hat{i} - (-2-(-4))\hat{j} + (1-2)\hat{k}$$

$$= \langle 0,-2,-1 \rangle$$

So, L_2: $\vec{r} = \langle 3,4,0 \rangle + s\langle 0,-2,-1 \rangle = \langle 3, 4-2s, -s \rangle$

Finally, the distance between two lines is given by

$$d = \frac{|\vec{PP} \cdot (\vec{v}_1 \times \vec{v}_2)|}{\|\vec{v}_1 \times \vec{v}_2\|}$$

where $\vec{PP}$ is a vector joining a point on L_1 with a point on L_2 and $\vec{v}_1$ and $\vec{v}_2$ are the direction vectors of L_1 and L_2.

$\vec{PP} = \langle 3-1, 4-2, 0-6 \rangle = \langle 2, 2, -6 \rangle$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 0 & -2 & -1 \end{vmatrix} = (-2-(-4))\hat{i} - (-1-0)\hat{j} + (-2-0)\hat{k} = \langle 2, 1, -2 \rangle$$

$$\|\vec{v}_1 \times \vec{v}_2\| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$d = \frac{|\langle 2,2,-6\rangle \cdot \langle 2,1,-2\rangle|}{3} = \frac{|4+2+12|}{3} =$$

$$\frac{18}{3} = \underline{6}|$$

7. Report

8. $x^2+y^2=4$ is a circle with center at $(0,0)$ and radius 2. We may use then the parametrization $x = 2\cos t$, $y = 2\sin t$.

Now, since $z = xy = 4\sin t\cos t$.

Thus, the curve of intersection is

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\sin t\cos t \rangle$$

9. $\vec{r}(t) = \langle e^t, te^t, te^{t^2} \rangle$

$$\vec{r}\,'(t) = \langle e^t, e^t + te^t, e^{t^2} + te^{t^2}(2t) \rangle$$

$$= \langle e^t, e^t(1+t), e^{t^2}(1+2t^2) \rangle$$

at $(1,0,0)$ $t=0$, thus

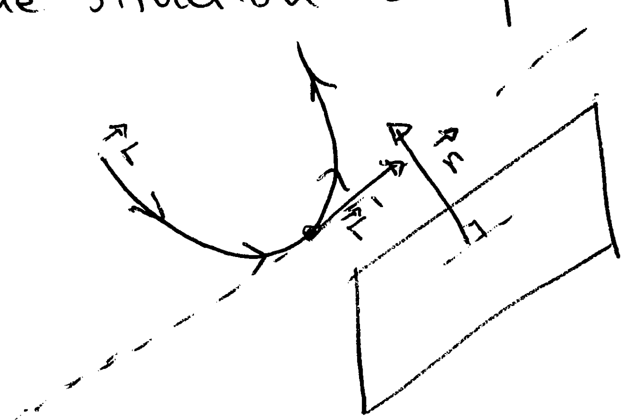
$$\vec{r}\,'(0) = \langle 1, 1, 1 \rangle$$

Thus, the tangent line is given by

$$T: \langle 1,0,0 \rangle + t\langle 1,1,1 \rangle = \langle 1+t, t, t \rangle$$

$$x = 1+t, \quad y = t, \quad z = t$$

10. The situation is depicted below:



The tangent line will be parallel to the plane when the line's direction vector is perpendicular to the plane's normal vector. Since the line's direction vector is the space curve's derivative vector, we have $\vec{r}\,'(t) \cdot \vec{n} = 0$

So
$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, e^t \rangle$$
$$\vec{n} = \langle \sqrt{3}, 1, 0 \rangle$$

so
$$\langle -2\sin t, 2\cos t, e^t \rangle \cdot \langle \sqrt{3}, 1, 0 \rangle =$$
$$-2\sqrt{3}\sin t + 2\cos t = 0$$
$$\Rightarrow \quad 2\cos t = 2\sqrt{3}\sin t$$
$$1 = \sqrt{3}\tan t \Rightarrow$$
$$t = \arctan\left(\frac{1}{\sqrt{3}}\right) = 0.5723$$

Therefore, the point at which the tangent line is parallel to the plane is

$$\left(2\cos(0.5723),\ 2\sin(0.5723),\ e^{0.5723}\right)$$