Homework # 4

Math 243 - Section 51

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1. The length of $\vec{r}(t)$ can be found through

$$s = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

In this case, $a=0$, $b=\frac{\pi}{4}$.

$$\vec{r}\,'(t) = \left\langle -\sin t, \cos t, \frac{1}{\cos t}(-\sin t) \right\rangle$$

$$= \langle -\sin t, \cos t, -\tan t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$$

$$= \sqrt{1 + \tan^2 t}$$

$$= \sqrt{\sec^2 t} = |\sec t| = \sec t, \quad 0 \le t \le \frac{\pi}{4}$$

$$s = \int_0^{\pi/4} \sec t \, dt = \ln \left| \sec t + \tan t \right| \Big|_0^{\pi/4}$$

$$= \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \ln \left| \sec 0 + \tan 0 \right|$$

$$= \ln \left| \sqrt{2} + 1 \right| - \ln |1|$$

$$= \ln (\sqrt{2} + 1)$$

2. $$\kappa(t) = \frac{\| \vec{r}\,'(t) \times \vec{r}\,''(t) \|}{\| \vec{r}\,'(t) \|^3}$$

$$\vec{r}\,'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$$

$$\vec{r}\,''(t) = \langle 0, e^t, e^{-t} \rangle$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sqrt{2} & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = (1+1)\hat{\imath} - (\sqrt{2}e^{-t})\hat{\jmath} + (\sqrt{2}e^{t})\hat{k}$$

$$= \langle 2, -\sqrt{2}e^{-t}, \sqrt{2}e^{t} \rangle$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{4 + 2e^{-2t} + 2e^{2t}} = \sqrt{2}\sqrt{2 + e^{2t} + e^{-2t}}$$

$$\|\vec{r}'(t)\| = \sqrt{2 + e^{2t} + e^{-2t}}$$

$$\kappa(t) = \frac{\sqrt{2}\sqrt{2+e^{2t}+e^{-2t}}}{\left(\sqrt{2+e^{2t}+e^{-2t}}\right)^3} = \boxed{\frac{\sqrt{2}}{2+e^{2t}+e^{-2t}}}$$

3. $\vec{r}(x) = \langle x, \sin x, 0 \rangle \Rightarrow$

$$\vec{r}'(x) = \langle 1, \cos x, 0 \rangle$$

$$\vec{r}''(x) = \langle 0, -\sin x, 0 \rangle$$

$$\vec{r}'(x) \times \vec{r}''(x) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \cos x & 0 \\ 0 & -\sin x & 0 \end{vmatrix} = (0)\hat{i} - (0)\hat{j} + (-\sin x)\hat{k} =$$

$$\langle 0, 0, -\sin x \rangle$$

$$\|\vec{r}'(x) \times \vec{r}''(x)\| = \sqrt{\sin^2 x} = |\sin x|$$

$$\|\vec{r}'(x)\| = \sqrt{1 + \cos^2 x}$$

$K(x) = \dfrac{|\sin x|}{\sqrt{1+\cos^2 x}} \Rightarrow$ $K(x)$ is minimum when $|\sin x| = 0 \Rightarrow K(x)$ is minimum at $0, \pm\pi, \pm 2\pi, \pm 3\pi$ ...

4. Say $y=f(x)$, then $\vec{r}(t)=\langle t, f(t), 0\rangle$.

$\vec{r}\,'(t)=\langle 1, f'(t), 0\rangle$

$\vec{r}\,''(t)=\langle 0, f''(t), 0\rangle$

$$\vec{r}\,'(t)\times\vec{r}\,''(t)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & f'(t) & 0\\ 0 & f''(t) & 0\end{vmatrix}=0\hat{i}-0\hat{j}+f''(t)\hat{k}$$

$$\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|=\sqrt{(f''(t))^2}=|f''(t)|$$

$$\|\vec{r}\,'(t)\|=\sqrt{1+(f'(t))^2}\implies$$

$$\kappa(t)=\frac{|f''(t)|}{(1+f'(t))^{3/2}}\qquad(1)$$

At an inflection point $f''(t)=0 \implies \kappa(t)=0$ according to (1). Thus, at an inflection point, the curvature of $f(x)$ is zero.

5. $\vec{a}(t) = \langle 2, 6t, 12t^2 \rangle$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 2t, 3t^2, 4t^3 \rangle + \vec{C}$$

Since $\vec{v}(0) = \hat{\imath}$:

$$\vec{v}(0) = \langle 0, 0, 0 \rangle + \vec{C} = \hat{\imath} \Rightarrow \vec{C} = \hat{\imath}$$

So, $\underline{\vec{v}(t) = \langle 2t+1, 3t^2, 4t^3 \rangle}$

Now,

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle t^2+t, t^3, t^4 \rangle + \vec{D}$$

Since $\vec{r}(0) = \hat{\jmath} - \hat{k}$:

$$\vec{r}(0) = \langle 0, 0, 0 \rangle + \vec{D} = \hat{\jmath} - \hat{k}$$

we have

$$\underline{\vec{r}(t) = \langle t^2+t, t^3+1, t^4-1 \rangle}$$

6. $\vec{r}(t) = \langle t^2 - t, \cos t, \sin t \rangle$

$\vec{r}\,'(t) = \langle 2t - 1, -\sin t, \cos t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(2t-1)^2 + \sin^2 t + \cos^2 t}$

$= \sqrt{(2t-1)^2 + 1}$

The speed will be minimum at t_m such that

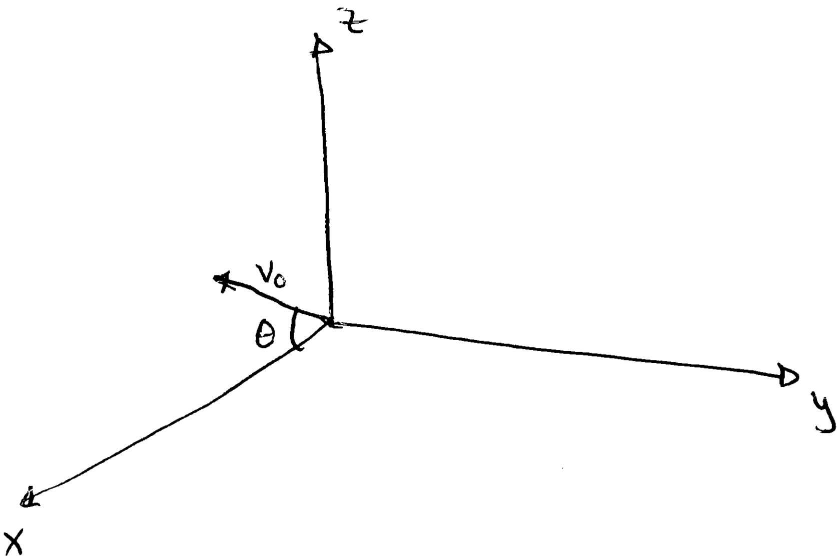
$$\left.\frac{d}{dt}\|\vec{r}\,'(t)\|\right|_{t_m} = 0$$

$$\frac{d}{dt}\sqrt{(2t-1)^2+1} = \frac{1}{2}\left((2t-1)^2+1\right)^{-\frac{1}{2}}\left(2(2t-1)\right)(2)$$

$$= \frac{4t-2}{\sqrt{(2t-1)^2+1}} \Rightarrow t_m = \frac{1}{2}$$

t_m is the point in time at which $\|\vec{r}\,'(t)\|$ is minimum because for $t < t_m$ $\frac{d}{dt}\|\vec{r}\,'(t)\| < 0$ and for $t > t_m$ $\frac{d}{dt}\|\vec{r}\,'(t)\| > 0$

7. Assuming that only the gravitational acceleration is acting on the projectile, then $\vec{a}(t) = \langle 0, 0, -g \rangle$. If the initial position of the particle is at the origin, then



where $\vec{v}(0) = \langle v_0 \cos\theta, 0, v_0 \sin\theta \rangle$, v_0 is the initial speed and θ is the angle at which the projectile is shot with respect to the horizon.

So,

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 0, 0, -gt \rangle + \vec{c}$$

Since $\vec{v}(0) = \langle v_0 \cos\theta, 0, v_0 \sin\theta \rangle$, then

$$\vec{c} = \langle v_0 \cos\theta, 0, v_0 \sin\theta \rangle$$

So $\vec{v}(t) = \langle v_0 \cos\theta, 0, v_0 \sin\theta - gt \rangle$

Then

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle v_0 \cos\theta\, t,\ 0,\ v_0 \sin\theta\, t - \frac{g}{2} t^2 \rangle + \vec{D}$$

Since $\vec{r}(0) = \vec{0}$, $\vec{D} = \vec{0}$ and

$$\vec{r}(t) = \langle v_0 \cos\theta\, t,\ 0,\ v_0 \sin\theta\, t - \frac{g}{2} t^2 \rangle$$

The maximum height is reached when the z-component of the velocity is zero, thus

$$v_0 \sin\theta - g t_m = 0 \Rightarrow t_m = \frac{v_0 \sin\theta}{g}$$

Therefore, the maximum height reached by the projectile is equal to the z-component of the position vector function evaluated at t_m:

$$h_m = v_0 \sin\theta\, t_m - \frac{g}{2} t_m^2$$

$$= v_0 \sin\theta \left(\frac{v_0 \sin\theta}{g}\right) - \frac{g}{2}\left(\frac{v_0 \sin\theta}{g}\right)^2$$

$$= \frac{v_0^2 \sin^2\theta}{g} - \frac{v_0^2 \sin^2\theta}{2g} = \frac{v_0^2 \sin^2\theta}{2g}$$

Now

$$\frac{3}{4}h_m = \frac{3V_0^2\sin^2\theta}{8g}$$

This height is reached at some t_q. From the position vector function:

$$V_0\sin\theta\, t_q - \frac{g}{2}t_q^2 = \frac{3V_0^2\sin^2\theta}{8g}$$

multiplying by $8g$:

$$8g\,V_0\sin\theta\, t_q - 4g^2t_q^2 = 3V_0^2\sin^2\theta$$

Rewriting:

$$4g^2t_q^2 - 8g\,V_0\sin\theta\, t_q + 3V_0^2\sin^2\theta = 0$$

Using the quadratic formula:

$$t_q = \frac{8gV_0\sin\theta \pm\sqrt{64g^2V_0^2\sin^2\theta - 4(4g^2)(3V_0^2\sin^2\theta)}}{8g^2}$$

$$= \frac{V_0\sin\theta}{8g}(8\pm4)$$

This means that $\frac{3}{4}h_m$ is reached twice, which makes sense: one time on the way up and once more on the way down

So, focusing on the trajectory upward:

$$\boxed{t_q = \frac{V_0 \sin\theta}{8g}(4) = \frac{V_0 \sin\theta}{2g} = \frac{t_m}{2}}$$

8. The domain of $f(x,y) = \sqrt{xy}$ is the set of points (x,y) such that

$$xy \geq 0$$

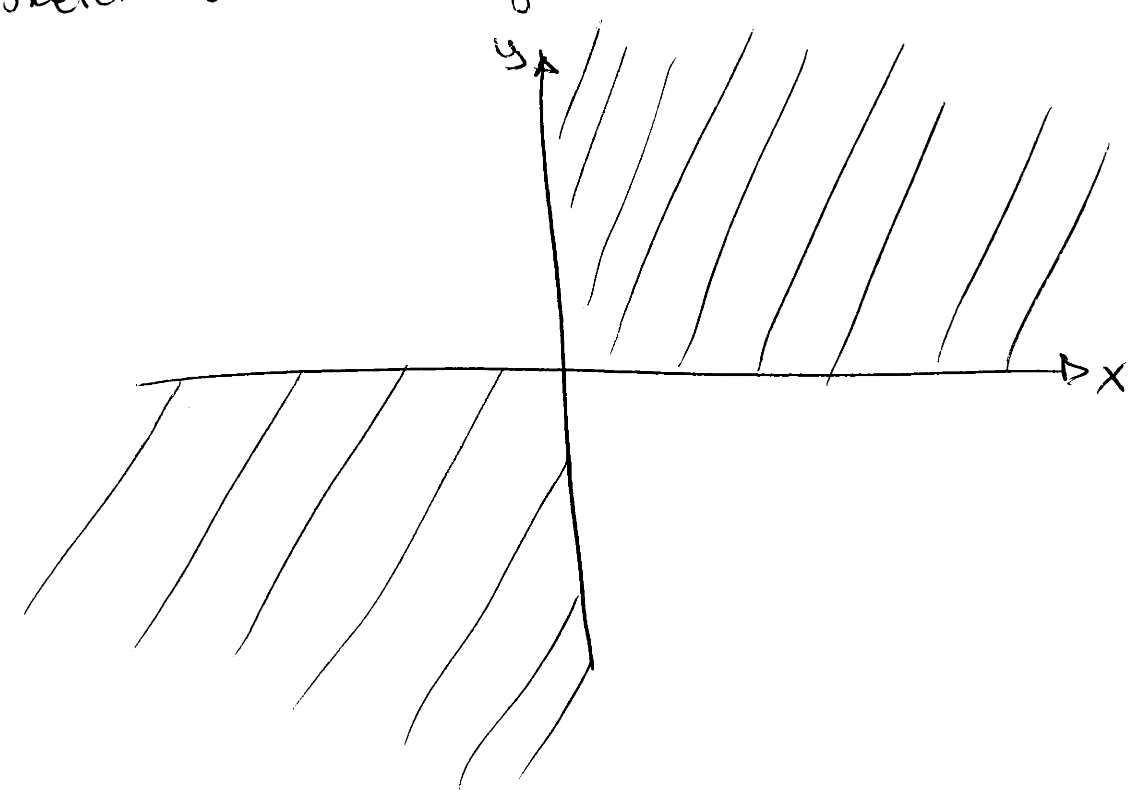
This region is

$$x \geq 0 \ \& \ y \geq 0$$

or

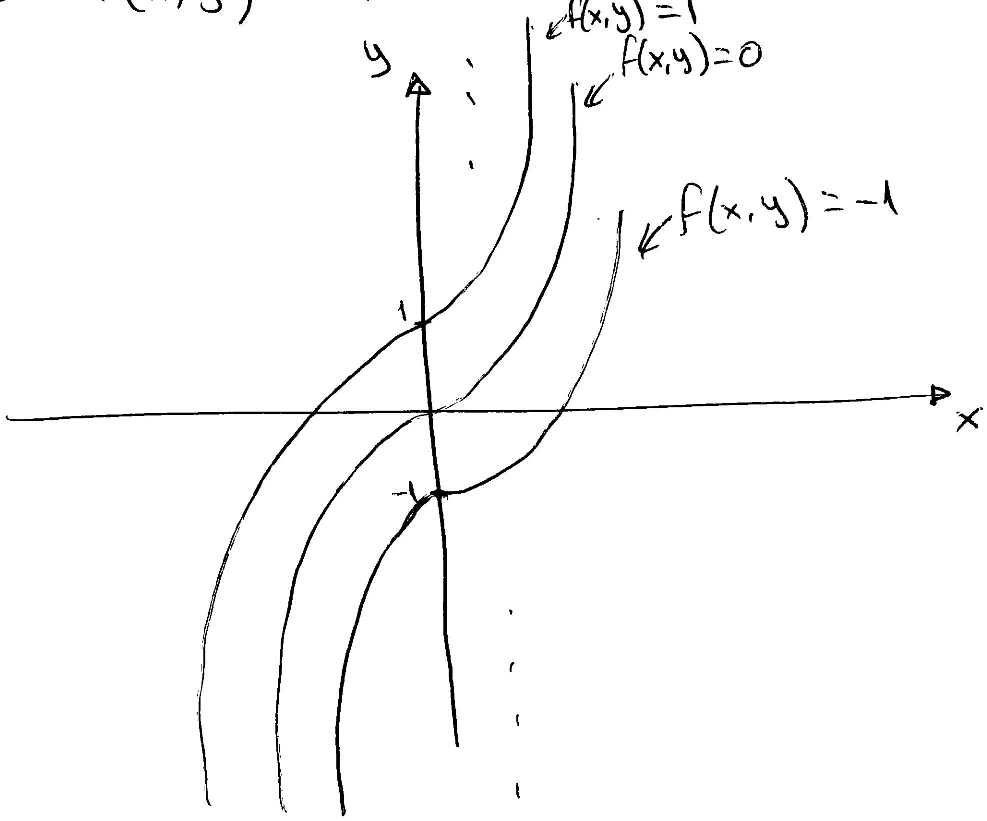
$$x \leq 0 \ \& \ y \leq 0$$

A sketch of this region is



9. $f(x,y) = x^3 - y$

So, for $f(x,y) = K$ we have



10. A level surface for $f(x,y,z) = (x-2)^2 + y^2 + z^2$ is a surface that satisfies $(x-2)^2 + y^2 + z^2 = K$. It is possible to show that such an equation describes a sphere centered at $(2,0,0)$ with radius $\sqrt{K}$. Thus, the level surfaces of $f(x,y,z)$ are concentric spheres of various radii centered at $(2,0,0)$.