Homework #5
Math 243 - Section 51
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1. Approaching $(1,0)$ through $x=1$:

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} = \lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2} =$$

$$\lim_{(x,y)\to(1,0)} \frac{0y}{0+y^2} = 0$$

Approaching $(1,0)$ through $y=x-1$:

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} = \lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2} =$$

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2}{2(x-1)^2} = \frac{1}{2}$$

$\therefore$ the limit does not exist.

2.

a) $f(x,y) = x^y$

$f_x = y\,x^{y-1}$

$f_y = (\ln x)\,x^y$

b) $F_\alpha = \dfrac{\partial}{\partial \alpha}\displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt = \dfrac{\partial}{\partial \alpha}\left(-\displaystyle\int_\beta^\alpha \sqrt{t^3+1}\,dt\right) =$

$-\sqrt{\alpha^3+1}$ by the FTC

$F_\beta = \dfrac{\partial}{\partial \beta}\displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt = \sqrt{\beta^3+1}$ also by the

FTC

c) $f(x,y,z) = x\sin(y-z)$

$f_x = \sin(y-z)$

$f_y = x\cos(y-z)(1) = x\cos(y-z)$

$f_z = x\cos(y-z)(-1) = -x\cos(y-z)$

3. a) $f(x,y) = \dfrac{xy}{x-y}$

$$f_x = \frac{(x-y)y - xy(1)}{(x-y)^2} = \frac{xy - y^2 - xy}{(x-y)^2} =$$

$$\frac{-y^2}{(x-y)^2}$$

$$f_y = \frac{(x-y)x - xy(-1)}{(x-y)^2} = \frac{x^2 - xy + xy}{(x-y)^2} = \frac{x^2}{(x-y)^2}$$

$$f_{xx} = -y^2(-2)(x-y)^{-3}(1) = \boxed{\frac{2y^2}{(x-y)^3}}$$

$$f_{xy} = \frac{(x-y)^2(-2y) - (-y^2)(2(x-y)(-1))}{(x-y)^4} =$$

$$\frac{-2y(x-y)^2 - 2y^2(x-y)}{(x-y)^4} =$$

$$\frac{-2(x-y)[y(x-y) + y^2]}{(x-y)^4} = \boxed{-\frac{2xy}{(x-y)^3}}$$

$$f_{yx} = \frac{(x-y)^2(2x) - x^2(2(x-y)(1))}{(x-y)^4} =$$

$$\frac{2x(x-y)^2 - 2x^2(x-y)}{(x-y)^4} =$$

$$\frac{2(x-y)[x(x-y) - x^2]}{(x-y)^4} = \boxed{-\frac{2xy}{(x-y)^3}}$$

$$f_{yy} = x^2(-2(x-y)^{-3}(-1)) = \boxed{\frac{2x^2}{(x-y)^3}}$$

b) $f(u,v) = \sqrt{u^2+v^2}$

$$f_u = \frac{1}{\cancel{2}}(u^2+v^2)^{-\frac{1}{2}}(\cancel{2}u) = \frac{u}{\sqrt{u^2+v^2}}$$

By symmetry:

$$f_v = \frac{v}{\sqrt{u^2+v^2}}$$

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$$f_{uu} = \frac{\sqrt{u^2+v^2}(1) - u\frac{1}{2}(u^2+v^2)^{-\frac{1}{2}}(2u)}{\left(\sqrt{u^2+v^2}\right)^2} =$$

$$\frac{\sqrt{u^2+v^2} - \dfrac{u^2}{\sqrt{u^2+v^2}}}{\left(\sqrt{u^2+v^2}\right)^2} =$$

$$\frac{\dfrac{\left(\sqrt{u^2+v^2}\right)^2 - u^2}{\sqrt{u^2+v^2}}}{\left(\sqrt{u^2+v^2}\right)^2} = \boxed{\frac{v^2}{\left(\sqrt{u^2+v^2}\right)^3}}$$

By symmetry:

$$f_{vv} = \frac{u^2}{\left(\sqrt{u^2+v^2}\right)^3}$$

$$f_{uv} = \frac{\left(\sqrt{u^2+v^2}\right)(0) - u\frac{1}{2}(u^2+v^2)^{-\frac{1}{2}}(2v)}{\left(\sqrt{u^2+v^2}\right)^2} =$$

$$\frac{\dfrac{-uv}{\sqrt{u^2+v^2}}}{\left(\sqrt{u^2+v^2}\right)^2} = \boxed{-\frac{uv}{\left(\sqrt{u^2+v^2}\right)^3}}$$

Again by symmetry:

$$f_{vu} = -\frac{uv}{\left(\sqrt{u^2+v^2}\right)^3}$$

4. Since x is constant, the slope of the tangent line is given by $\frac{\partial z}{\partial y} = -4y$. Thus an equation of the line on the plane $x=1$ is

$$z - z_0 = m(y - y_0)$$
$$z - (-4) = -4(2)(y-2)$$
$$z + 4 = -8(y-2)$$
$$z = -8y + 12$$

If we think of this equation as one of the symmetric equations of the line, then $x=1$, $z=t$, and $-8y+12=t$, therefore the parametric equations of the line are

$$\left.\begin{aligned} x &= 1 \\ y &= \frac{t-12}{-8} \\ z &= t \end{aligned}\right\}$$

5.

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Since $f(x,y) = \sqrt{xy}$, $f_x(x,y) = \frac{1}{2}(xy)^{-\frac{1}{2}}(y) = \dfrac{y}{2\sqrt{xy}}$

$$f_y(x,y) = \frac{1}{2}(xy)^{-\frac{1}{2}}(x) = \frac{x}{2\sqrt{xy}}$$

Thus,

$$z - 1 = \frac{1}{2\sqrt{(1)(1)}}(x-1) + \frac{1}{2(\sqrt{(1)(1)})}(y-1)$$

$$\boxed{z - 1 = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)}$$

6. $f(x,y) = x^2 + xy + 3y^2$

$f_x(x,y) = 2x + y$

$f_y(x,y) = 6y + x$

An equation of the tangent plane is:

$$z - 5 = (2(1)+1)(x-1) + (6(1)+1)(y-1)$$

$$\boxed{z - 5 = 3(x-1) + 7(y-1)}$$

7. $f(x,y) = xe^{xy}$

$f_x(x,y) = e^{xy} + xe^{xy}(y) = e^{xy}(1 + xy)$

$f_y(x,y) = xe^{xy}(x) = x^2e^{xy}$

An equation of the tangent plane is:

$z - 2 = e^{2(0)}(1 + 2(0))(x-2) + (2)^2e^{2(0)}(y-0)$

$z - 2 = (x-2) + 4y = x + 4y - 2$

8. $f(x,y) = x^3y^4$

$f_x(x,y) = 3x^2y^4$

$f_y(x,y) = 4x^3y^3$

The linearization of $f(x,y)$ at $(\underset{x_0}{1}, \underset{y_0}{1})$ is

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$L(x,y) = 1 + 3(x-1) + 4(y-1)$

$= 3x + 4y + 1 - 3 - 4$

$= 3x + 4y - 6$

9. $f(x,y) = e^{-xy}\cos y$

$f_x(x,y) = \cos y\, e^{-xy}(-y) = -ye^{-xy}\cos y$

$f_y(x,y) = e^{-xy}(-x)\cos y + e^{-xy}(-\sin y)$

$= -xe^{-xy}\cos y - e^{-xy}\sin y$

$= -e^{-xy}(x\cos y + \sin y)$

$L(x,y) = e^{-\pi(0)}\cos 0 + (-0e^{-\pi(0)}\cos 0)(x-\pi) +$
$(-e^{-\pi(0)}(\pi\cos 0 + \sin 0))(y-0)$

$= 1 + 0 + (-\pi)y$

$= 1 - \pi y$

10. At $P(2,1,3)$:

$$\left.\begin{aligned} 2+3t &= 2 \\ 1-t^2 &= 1 \\ 3-4t+t^2 &= 3 \end{aligned}\right\} \Rightarrow t = 0$$

$$\left.\begin{aligned} 1+u^2 &= 2 \\ 2u^3-1 &= 1 \\ 2u+1 &= 3 \end{aligned}\right\} \Rightarrow u = 1$$

Tangent vectors to $\vec{r}_1(t)$ and $\vec{r}_2(u)$ are

$$\vec{r}_1'(t) = \langle 3, -2t, -4+2t \rangle$$

@ $t=0$

$$\vec{r}_1'(0) = \langle 3, 0, -4 \rangle = \vec{a}$$

$$\vec{r}_2'(u) = \langle 2u, 6u^2, 2 \rangle$$

@ $u=1$

$$\vec{r}_2'(1) = \langle 2, 6, 2 \rangle = \vec{b}$$

A normal vector to the tangent plane at P is

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix} = (0-(-24))\hat{i} - (6+8)\hat{j} + (18-0)\hat{k} =$$

$$\langle 24, -14, 18 \rangle$$

$\therefore$ An equation of the tangent plane is

$$\boxed{24(x-2) - 14(y-1) + 18(z-3) = 0}$$