Homework #6
Math 243 - Section 51
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1. $z = \sin^2(xy)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= 2\sin(xy)(\cos(xy))(y)\, dx + 2\sin(xy)(\cos(xy))(x)\, dy$$

$$= 2y\sin(xy)\cos(xy)\, dx + 2x\sin(xy)\cos(xy)\, dy$$

$$= y\sin(2xy)\, dx + x\sin(2xy)\, dy$$

2. If V is the volume of the can, then $V = \pi r^2 h$. The amount of metal is roughly

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$= 2\pi r h\, dr + \pi r^2 dh$$

In our case, $h = 4$, $r = 1$, $dr = 0.01$, and $dh = 2(0.013) = 0.026$ (to consider top and bottom)

So,

$$dV = 2\pi(1)(4)0.01 + \pi(1)^2(0.026)$$

$$= 0.2513 + 0.0816$$

$$\approx 0.333 \text{ in}^3$$

3. $z = \arctan\left(\frac{y}{x}\right)$, $x = e^t$, $y = 1 - e^{-t}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)e^t + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}\left(e^{-t}\right)$$

$$= -\frac{y}{x^2+y^2}e^t + \frac{x}{x^2+y^2}e^{-t}$$

4. $z = e^{x-y^2}$, $x = ts^2$, $y = st^2$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial s} = \left(e^{x-y^2}\right)(1)(2ts) + \left(e^{x-y^2}\right)(-2y)(t^2)$$

$$= \underline{2ts\, e^{x-y^2} - 2yt^2 e^{x-y^2}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left(e^{x-y^2}\right)(1)(s^2) + \left(e^{x-y^2}\right)(-2y)(2ts)$$

$$= \underline{s^2 e^{x-y^2} - 4yst\, e^{x-y^2}}$$

5. $T(x,y) = 4x + 3y + 1$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

$$= 4\frac{d}{dt}\sqrt{1+t} + 3\frac{d}{dt}\left(2 + \frac{t}{3}\right)$$

$$= 4\,\frac{1}{2}(1+t)^{-\frac{1}{2}} + 3\left(\frac{1}{3}\right)$$

$$= 2(1+t)^{-\frac{1}{2}} + 1$$

$$\left.\frac{dT}{dt}\right|_{t=3} = 2\left(\frac{1}{\sqrt{1+3}}\right) + 1 = \underline{2\ {}^{\circ}C/s}$$

6. $\vec{v} = \langle -1, 1 \rangle$

$\hat{v} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$$D_{\hat{v}} f(x,y) = \nabla f(x,y) \cdot \hat{v}$$

$$= \langle ye^{-x}(-1), e^{-x} \rangle \cdot \hat{v}$$

$$= \langle -ye^{-x}, e^{-x} \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= \frac{ye^{-x}}{\sqrt{2}} + \frac{e^{-x}}{\sqrt{2}}$$

$$D_{\hat{v}} f(0,4) = \frac{4e^{0}}{\sqrt{2}} + \frac{e^{0}}{\sqrt{2}} = \boxed{\frac{5}{\sqrt{2}}}$$

7. $f(x,y,z) = y^3 e^{xyz}$

$$\nabla f(x,y,z) = \langle y^3 e^{xyz}(yz), 3y^2 e^{xyz} + y^3 e^{xyz}(xz), y^3 e^{xyz}(xy) \rangle$$

$$= \langle y^4 z e^{xyz}, y^2 e^{xyz}(3 + xyz), xy^4 e^{xyz} \rangle$$

$$\nabla f(0,1,-1) = \langle 1^4(-1)e^0, 1^2 e^0(3+0), 0\rangle$$
$$= \boxed{\langle -1, 3, 0\rangle}$$

$$D_{\hat{v}} f(x,y,z) = \nabla f(x,y,z)\cdot \hat{v}$$

$\vec{v} = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$, since $\|\vec{v}\| = 1$, we can use $\vec{v}$ directly, thus:

$$D_{\vec{v}} f(0,1,-1) = \langle -1, 3, 0\rangle \cdot \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$$
$$= -\frac{3}{13} + \frac{12}{13} = \boxed{\frac{9}{13}}$$

8. The maximum rate of change of $f(x,y) = \sin(xy)$ at $(1,0)$ is equal to $\|\nabla f(1,0)\|$ and the direction in which it occurs is $\nabla f(1,0)$.

So $\nabla f(x,y) = \langle \cos(xy)(y), \cos(xy)(x)\rangle$
$$= \langle y\cos(xy), x\cos(xy)\rangle$$

$\nabla f(1,0) = \langle 0, \cos 0 \rangle = \underline{\langle 0, 1 \rangle}$

$\|\nabla f(1,0)\| = \underline{1}$

9. $f(x,y) = ye^{-xy}$

$\nabla f(x,y) = \langle ye^{-xy}(-y), e^{-xy} + ye^{-xy}(-x) \rangle$

$= \langle -y^2e^{-xy}, e^{-xy}(1-xy) \rangle$

$\nabla f(0,2) = \langle -4e^0, e^0(1-0) \rangle$

$= \langle -4, 1 \rangle$

Let $\hat{u} = \langle u_x, u_y \rangle$

So $D_{\hat{u}} f(0,2) = \nabla f(0,2) \cdot \langle u_x, u_y \rangle$

$= \langle -4, 1 \rangle \cdot \langle u_x, u_y \rangle$

$= -4u_x + u_y = 1$ ①

Since $\|\hat{u}\| = 1 \Rightarrow u_x^2 + u_y^2 = 1$ ②

From 1: $u_y = 1 + 4u_x$

In 2:

$$u_x^2 + (1+4u_x)^2 = 1$$

$$u_x^2 + 1 + 8u_x + 16u_x^2 = 1$$

$$17u_x^2 + 8u_x = 0$$

$$u_x(17u_x + 8) = 0 \Rightarrow \begin{array}{l} u_x = 0 \text{ or} \\ u_x = -\frac{8}{17} \end{array} \Rightarrow$$

$$u_y = 1 \text{ or}$$

$$u_y = 1 - \frac{32}{17}$$

$$= \frac{17-32}{17} = -\frac{15}{17}$$

So

$$\hat{u} = \langle 0, 1 \rangle \text{ or}$$

$$\hat{u} = \left\langle -\frac{8}{17}, -\frac{15}{17} \right\rangle$$

10. If the ellipsoid and the sphere are tangent to each other at $(1,1,2)$, the normal vectors of the tangent planes at $(1,1,2)$ must be parallel.

A vector normal to the ellipsoid $f_1(x,y,z) = 3x^2+2y^2+z^2=9$ is $\nabla f_1(x,y,z) = \langle 6x, 4y, 2z\rangle$ which at $(1,1,2)$ is $\nabla f_1(1,1,2) = \langle 6,4,4\rangle$.

A vector normal to the sphere $f_2(x,y,z) = x^2+y^2+z^2-8x-6y-8z-24=0$ is $\nabla f_2(x,y,z) = \langle 2x-8, 2y-6, 2z-8\rangle$ which at $(1,1,2)$ is $\nabla f_2(1,1,2) = \langle -6,-4,-4\rangle$.

Since $\nabla f_1(1,1,2) = -\nabla f_2(1,1,2)$
$\nabla f_1(1,1,2)$ is parallel to $\nabla f_2(1,1,2)$ and the two surfaces are tangent to each other at $(1,1,2)$.