Homework #7

Math 243 - Section 51

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1. $f(x,y) = x^2y - xy - 2x + 3$

$\nabla f(x,y) = \langle 2xy - y - 2,\ x^2 - x \rangle$

For critical points (x^*, y^*), $\nabla f(x^*, y^*) = \vec{0}$

So:

$2x^*y^* - y^* - 2 = 0 \quad (1)$

$x^{*2} - x^* = 0 \quad (2)$

From (2):

$x^*(x^* - 1) = 0 \Rightarrow x^* = 0$ or $x^* = 1$

In (1):

If $x^* = 0$, $-y^* - 2 = 0 \Rightarrow y^* = -2$

If $x^* = 1$, $2y^* - y^* - 2 = 0$

$y^* - 2 = 0 \Rightarrow y^* = 2$

So, the critical points are: $(0,-2)$ and $(1,2)$

Now, the Hessian of f is:

$$Hf(x,y) = \begin{pmatrix} 2y & 2x-1 \\ 2x-1 & 0 \end{pmatrix}$$

@ $(0,-2)$

$$Hf(0,-2) = \begin{pmatrix} -4 & -1 \\ -1 & 0 \end{pmatrix}$$

$\det(Hf(0,-2)) = -1 < 0$, so $(0,-2)$ is a saddle point.

@ $(1,2)$

$$Hf(1,2) = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$$

$\det(Hf(1,2)) = -1 < 0$, so $(1,2)$ is also a saddle point.

2. $f(x,y) = x^2y + 2x^2 + y^2$

$$\nabla f(x,y) = \langle 2xy + 4x,\ x^2 + 2y \rangle$$

Critical points satisfy $\nabla f(x^*, y^*) = \vec{0}$, so

$$2x^*y^* + 4x^* = 0 \qquad (1)$$
$$(x^*)^2 + 2y^* = 0 \qquad (2)$$

From (1): $2x^*y^* = -4x^*$ (assuming $x^* \neq 0$)

$$y^* = \frac{-4x^*}{2x^*} = -2$$

In (2):

$$(x^*)^2 + 2(-2) = 0$$
$$(x^*)^2 - 4 = 0 \Rightarrow (x^*)^2 = 4$$
$$x^* = \pm 2$$

Also from (1)

$2x^*(y^* + 2) = 0 \Rightarrow$ $x^* = 0$, and from (2) $y^* = 0$
or $y^* = -2$ (we have already found this solution)

So, the critical points are:
$(2,-2)$, $(-2,-2)$, $(0,0)$

The Hessian of f is

$$Hf(x,y) = \begin{pmatrix} 2y+4 & 2x \\ 2x & 2 \end{pmatrix}$$

@ $(0,0)$

$$Hf(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

$\det(Hf(0,0)) = 8 > 0$ and since $f_{xx} = 4 > 0$, $(0,0)$ is a local minimum.

@ $(2,-2)$

$$Hf(2,-2) = \begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix}$$

$\det(Hf(2,-2)) = -16 < 0$, so $(2,-2)$ is a saddle point.

@ $(-2,-2)$

$$Hf(-2,-2) = \begin{pmatrix} 0 & -4 \\ -4 & 2 \end{pmatrix}$$

$\det(Hf(-2,-2)) = -16 < 0$, so $(-2,-2)$ is also a saddle point.

3. $f(x,y) = x^2 + axy + y^2$

$\nabla f(x,y) = \langle 2x + ay,\ ax + 2y \rangle$

At a critical point (x^*, y^*) $\nabla f(x^*, y^*) = \vec{0}$, so

$2x^* + ay^* = 0 \quad (1)$

$2y^* + ax^* = 0 \quad (2)$

From (1): $x^* = -\frac{a}{2} y^*$

In (2): $2y^* + a\left(-\frac{a}{2} y^*\right) = 0$

$2y^* - \frac{a^2}{2} y^* = 0 \Rightarrow \left.\begin{array}{l} y^* = 0 \\ x^* = 0 \end{array}\right\}$ for any a

The Hessian is

$$Hf(x,y) = \begin{pmatrix} 2 & a \\ a & 2 \end{pmatrix}$$

$\det(Hf(x,y)) = 4 - a^2$

Case 1) $a=4$

$\det(Hf(0,0)) = 4-16 = -12 < 0$, so $(0,0)$ is a saddle point

Case 2) $a=2$

$\det(Hf(0,0)) = 4-4 = 0$, so $(0,0)$ may be a local maximizer, a local minimizer or a saddle point.

Case 3) $a=1$

$\det(Hf(0,0)) = 4-1 = 3 > 0$ and since $f_{xx} = 2 > 0$ $(0,0)$ is a local minimizer.

The coefficient a changes the curvature of the surface generated by $f(x,y)$ changing it in such a way that can be a saddle point, a point on a flat region, or a minimizer.

4. $f(x,y) = x^2y^2 - 2xy + 2x + y^2$

$\nabla f(x,y) = \langle 2xy^2 - 2y + 2, \ 2x^2y - 2x + 2y \rangle$

If $\nabla f(x^*, y^*) = \vec{0}$

$$\left.\begin{aligned} 2x^*(y^*)^2 - 2(y^*) + 2 &= 0 \\ 2(x^*)^2 y^* - 2(x^*) + 2(y^*) &= 0 \end{aligned}\right\}$$

This system of equations does not have real solutions, which means that $f(x,y)$ does not have critical points.

5. $f(x,y) = x^3y + x^2 + y^2$, $D = [0,1] \times [0,1]$

$\nabla f(x,y) = \langle 3x^2y + 2x, \ x^3 + 2y \rangle$

If $\nabla f(x^*, y^*) = \vec{0}$, then

$3(x^*)^2 y^* - 2x^* = 0 \qquad (1)$

$(x^*)^3 + 2y^* = 0 \qquad (2)$

From (2): $y^* = -\dfrac{(x^*)^3}{2}$

In (1):

$$3(x^*)^2\left(-\frac{(x^*)^3}{2}\right) - 2x^* = 0$$

$$-\frac{3}{2}(x^*)^5 - 2x^* = 0$$

$$x^*\left(\frac{3}{2}(x^*)^4 - 2\right) = 0 \Rightarrow \begin{array}{l} x^* = 0 \quad \text{or} \\ \frac{3}{2}(x^*)^4 - 2 = 0 \end{array}$$

$$3(x^*)^4 = 4$$

$$(x^*)^4 = \frac{4}{3}$$

$$x^* = \pm\sqrt[4]{\frac{4}{3}}$$

This solution is outside D, so we discard it.

So, the only critical point inside D is $(0,0)$.

Let us now find the critical points along the boundary of D.

$x=0$:

$f(0,y) = y^2 \Rightarrow$ $y=0$ is a minimum along $x=0$

$y=1$:

$\phi(x) = f(x,1) = x^3 + x^2 + 1$

$\phi'(x) = 3x^2 + 2x = 0 \Rightarrow$ $x=0$, or $3x+2=0$

$x = -\frac{2}{3} < 0$ so it is out of D.

$x=1$:

$\phi(y) = f(1,y) = y + 1 + y^2$

$\phi'(y) = 1 + 2y = 0 \Rightarrow y = -\frac{1}{2} < 0$, so it's out of D.

$y=0$:

$\phi(x) = f(x,0) = x^2 \Rightarrow$ $x=0$ is a minimum along $y=0$.

So, the points that qualify as potential absolute maximum and minimum are:

$(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$

$f(0,0) = 0$ ← absolute minimum

$f(0,1) = 1$

$f(1,1) = 3$ ← absolute maximum

$f(1,0) = 1$

6. $f(x,y) = e^{xy}$ subject to $\underbrace{x^3 + y^3}_{g(x,y)} = 16$

$$\nabla f(x,y) = \lambda g(x,y)$$

$$\langle e^{xy} y, e^{xy} x \rangle = \lambda \langle 3x^2, 3y^2 \rangle$$

$$y e^{xy} = 3\lambda x^2 \quad (1)$$

$$x e^{xy} = 3\lambda y^2 \quad (2)$$

$$x^3 + y^3 = 16 \quad (3)$$

From (1): $\lambda = \dfrac{y e^{xy}}{3x^2}$

In (2): $x e^{xy} = 3\left(\dfrac{y e^{xy}}{3x^2}\right) y^2$

$$x e^{xy} = \frac{y^3 e^{xy}}{x^2}$$

$$x^3 e^{xy} = y^3 e^{xy} \Rightarrow x^3 = y^3 \quad \text{since } e^{xy} > 0$$

So, from (3)

$$2x^3 = 16 \Rightarrow x^3 = 8 \Rightarrow x = 2 \text{ and } y = 2$$

So, $f(2,2) = e^4$ is an extreme value of f along $x^3 + y^3 = 16$.

7. $f(x,y,z) = 3x - y - 3z$, subject to $\underbrace{x + y - z}_{g(x,y,z)} = 0$, $\underbrace{x^2 + 2z^2}_{h(x,y,z)} = 1$.

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$

$$\langle 3, -1, -3 \rangle = \lambda \langle 1, 1, -1 \rangle + \mu \langle 2x, 0, 4z \rangle$$

$3 = \lambda + 2x\mu \quad (1)$

$-1 = \lambda \quad (2)$

$-3 = -\lambda + 4z\mu \quad (3)$

(2) in (1) and (3):

$2x\mu = 3 - (-1) = 4$

$x\mu = 2 \quad (4)$

$4z\mu = -3 + (-1) = -4$

$z\mu = -1 \quad (5)$

(4) and (5) in $x^2+2z^2=1$

$$\left(\frac{2}{\mu}\right)^2+2\left(\frac{-1}{\mu}\right)^2=1$$

$$\frac{4}{\mu^2}+\frac{2}{\mu^2}=1$$

$$\frac{6}{\mu^2}=1 \Rightarrow \mu^2=6 \Rightarrow \mu=\pm\sqrt{6}$$

Therefore:

$$x=\pm\frac{2}{\sqrt{6}}$$

$$z=\mp\frac{1}{\sqrt{6}}$$

x and z in $x+y-z=0$

$$\frac{2}{\sqrt{6}}+y-\left(-\frac{1}{\sqrt{6}}\right)=0$$

$$\frac{3}{\sqrt{6}}+y=0 \Rightarrow y=-\frac{3}{\sqrt{6}}$$

or

$$-\frac{2}{\sqrt{6}}+y-\left(\frac{1}{\sqrt{6}}\right)=0$$

$$-\frac{3}{\sqrt{6}}+y=0 \Rightarrow y=+\frac{3}{\sqrt{6}}$$

$$f\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = 3\left(\frac{2}{\sqrt{6}}\right) + \frac{3}{\sqrt{6}} + \frac{3}{\sqrt{6}} = \frac{12}{\sqrt{6}}$$

$$f\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = 3\left(-\frac{2}{\sqrt{6}}\right) - \frac{3}{\sqrt{6}} - \frac{3}{\sqrt{6}} = -\frac{12}{\sqrt{6}}$$

8. $f(x,y) = x^2 + y^2 + 4x - 4y$ subject to $\underbrace{x^2 + y^2}_{g(x,y)} \leq 9$.

Along the boundary:

$$\nabla f(x,y) = \langle 2x+4, 2y-4 \rangle$$

$$\nabla g(x,y) = \langle 2x, 2y \rangle$$

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

$$\langle 2x+4, 2y-4 \rangle = \lambda \langle 2x, 2y \rangle$$

$$\begin{cases} 2x+4 = 2\lambda x & (1) \\ 2y-4 = 2\lambda y & (2) \\ x^2 + y^2 = 9 & (3) \end{cases}$$

From (1):

$$2x - 2\lambda x + 4 = 0$$

$$2x(1-2\lambda) = -4 \Rightarrow x = \frac{-2}{1-2\lambda}$$

From (2)

$$y = \frac{2}{1-2\lambda}$$

x and y in (3):

$$\frac{8}{(1-2\lambda)^2} = 9 \Rightarrow \frac{8}{9} = (1-2\lambda)^2$$

$$\pm\frac{2\sqrt{2}}{3} = 1-2\lambda \Rightarrow$$

$$\underline{\lambda = \frac{1}{2} \mp \frac{\sqrt{2}}{3}}$$

Back in x:

$$x = \frac{-2}{1-2\left(\frac{1}{2} \mp \frac{\sqrt{2}}{3}\right)} = \frac{-2}{1-1 \pm \frac{2\sqrt{2}}{3}} = \underline{\mp\frac{3}{\sqrt{2}}}$$

$$\therefore\ \underline{y = \pm\frac{3}{\sqrt{2}}}$$

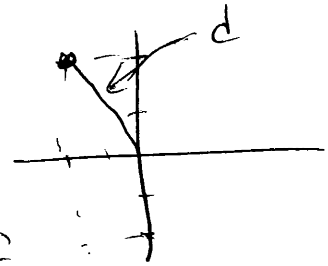
$$\left.\begin{aligned} f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) &= 9 - 12\sqrt{2} \\ f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) &= 9 + 12\sqrt{2} \end{aligned}\right\} \text{Extreme values}$$

Interior points:

$$\nabla f(x,y) = \langle 2x+4,\ 2y-4 \rangle = 0$$

$$2x + 4 = 0 \quad (1)$$
$$2y - 4 = 0 \quad (2)$$

From (1), $x = -2$
From (2), $y = 2$
$\Big\}$ Since $d < 3$ ← radius of region of interest, we take this point into account.



$$f(-2,2) = 4 + 4 - 8 - 8 = -8 \Big\} \text{ Extreme value}$$

9. $f(x,y) = 2x^2 + 3y^2 - 4x - 5$ subject to $\underbrace{x^2+y^2}_{g(x,y)} \leq 16$

Along the border:

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

$$\langle 4x-4,\ 6y \rangle = \lambda \langle 2x,\ 2y \rangle$$

$$\begin{cases} 4x - 4 = 2\lambda x & (1) \\ 6y = 2\lambda y & (2) \\ x^2 + y^2 = 16 & (3) \end{cases}$$

From (2):

$$6y - 2\lambda y = 0$$

$$2y(3-\lambda) = 0 \Rightarrow \begin{array}{l} y = 0 \quad \text{or} \\ \lambda = 3 \end{array}$$

If $y=0$, from (3):

$$x^2 = 16 \Rightarrow x = \pm 4$$

If $\lambda = 3$, from (1):

$$4x - 6x = 4$$

$$-2x = 4$$

$$x = -2$$

x in (3):

$$4 + y^2 = 16$$

$$y^2 = 12$$

$$y = \pm 2\sqrt{3}$$

So, critical points along $x^2+y^2=16$ are

$(\pm 4, 0)$ and $(-2, \pm 2\sqrt{3})$

$$\left.\begin{aligned} f(-4,0) &= 43 \\ f(4,0) &= 11 \\ f(-2,-2\sqrt{3}) &= 47 \\ f(-2,2\sqrt{3}) &= 47 \end{aligned}\right\} \text{ Extreme values}$$

Interior:

$$\nabla f(x,y) = \langle 4x-4,\ 6y \rangle = \vec{0}$$

$$\Rightarrow \quad x = 1,\ y = 0$$

$$f(1,0) = -7 \ \} \text{ Extreme value}$$

10. The problem can be cast as

$$\begin{aligned} &\max/\min \quad z \\ &\text{subject to} \quad 4x - 3y + 8z = 5 \\ &\qquad\qquad\quad -z^2 + x^2 + y^2 = 0 \end{aligned}$$

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$

$$\langle 0,0,1 \rangle = \lambda \langle 4,-3,8 \rangle + \mu \langle 2x, 2y, -2z \rangle$$

$$\left.\begin{aligned}
0 &= 4\lambda + 2\mu x \\
0 &= -3\lambda + 2\mu y \\
1 &= 8\lambda - 2\mu z \\
4x - 3y + 8z &= 5 \\
x^2 + y^2 - z^2 &= 0
\end{aligned}\right\}$$

Solving this system, one obtains:

MIN

$$\lambda = \frac{1}{13}, \quad \mu = -\frac{1}{2}, \quad x = \frac{4}{13}, \quad y = -\frac{3}{13}, \quad z = \frac{5}{13}$$

or

$$a = \frac{1}{3}, \quad \mu = \frac{1}{2}, \quad x = -\frac{4}{3}, \quad y = 1, \quad z = \frac{5}{3}$$

MAX