Homework # 8

Math 243 - Section 51

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1. $$\iint_R k\,dA = \int_c^d \int_a^b k\,dx\,dy = \int_c^d kx\Big|_a^b dy =$$

$$\int_c^d k(b-a)\,dy = k(b-a)y\Big|_c^d = \underline{k(b-a)(d-c)}$$

2. $$\int_0^1 \int_0^3 e^{x+3y}\,dx\,dy \; ; \; \begin{array}{l} u = x+3y \\ du = dx \end{array} \quad (\text{wrt } x)$$

$$\Rightarrow \int_0^1 \int_0^3 e^{x+3y}\,dx\,dy = \int_0^1 \int_{3y}^{3+3y} e^u\,du\,dy = \int_0^1 \left(e^{3+3y} - e^{3y}\right) dy =$$

$$\int_0^1 e^{3y}(e^3-1)\,dy = (e^3-1)\int_0^1 e^{3y}\,dy = \frac{1}{3}(e^3-1)(e^3-1) =$$

$$= \frac{1}{3}(e^3-1)^2$$

3. $\iint_R \frac{x}{1+xy}\, dA$, $R = [0,1] \times [0,1]$

$$\iint_R \frac{x}{1+xy}\, dA = \int_0^1\int_0^1 \frac{x}{1+xy}\, dy\, dx \qquad \begin{array}{l} u = 1+xy \\ du = x\, dy \ (\text{wrt } y) \end{array}$$

$$\int_0^1\int_1^{1+x} \frac{du}{u}\, dx = \int_0^1 \ln(u)\Big|_1^{1+x} dx = \int_0^1 (\ln(1+x) - \ln(1))\, dx$$

$$= \int_0^1 \ln(1+x)\, dx \ ; \qquad \begin{array}{l} u = 1+x \\ du = dx \end{array}$$

$$\Rightarrow \int_0^1 \ln(1+x)\, dx = \int_1^2 \ln u\, du = u\ln u - u \Big|_1^2 =$$

$$(2\ln(2) - 2) - (\ln(1) - 1) = 2\ln(2) - 2 - (0-1) =$$

$$\boxed{2\ln(2) - 1}$$

4. $g(x,y) = \int_a^x \int_c^y f(s,t)\,dt\,ds$, $a < x < b$
$c < y < d$

Let $\int_c^y f(s,t)\,dt = F(s,y) - F(s,c)$ by the FTC. Then

$$\int_a^x \left(F(s,y) - F(s,c)\right) ds = \mathcal{F}(x,y) - \mathcal{F}(a,y) - \left[\mathcal{F}(x,c) - \mathcal{F}(a,c)\right]$$

Again, by the FTC, where $\mathcal{F}$ is an antiderivative of F, and F is an antiderivative of f.

Thus,

$$g(x,y) = \underbrace{\mathcal{F}(x,y)}_{\text{function of } x,y} - \underbrace{\mathcal{F}(x,c)}_{\text{function of } x} - \underbrace{\mathcal{F}(a,y)}_{\text{function of } y} + \underbrace{\mathcal{F}(a,c)}_{\text{constant}}$$

Now,

$$g_x(x,y) = \mathcal{F}_x(x,y) - F(x,c) = F(x,y) - F(x,c)$$

$$g_{xy}(x,y) = F_y(x,y) = \underline{f(x,y)}$$

Similarly,

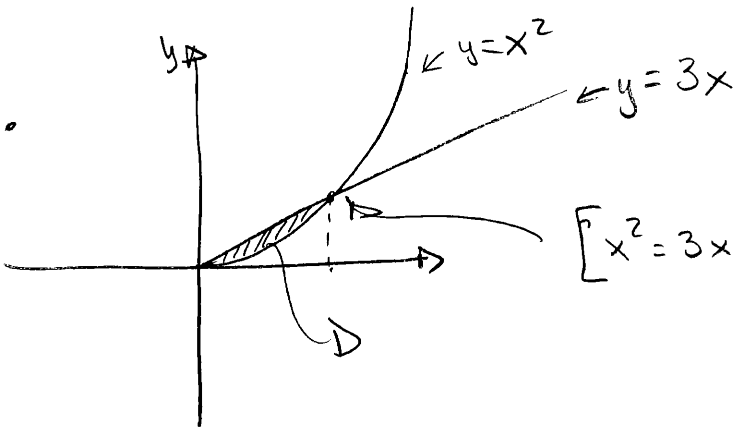
$$g_y(x,y) = \mathcal{F}_y(x,y) - F(a,y) = F(x,y) - F(a,y)$$

$$g_{yx}(x,y) = F_x(x,y) = \underline{f(x,y)}$$

5. $\displaystyle\int_0^2 \int_y^{2y} xy\, dx\, dy = \int_0^2 \frac{x^2}{2} y \Big|_y^{2y} dy = \int_0^2 \left(\frac{4y^3}{2} - \frac{y^3}{2}\right) dy =$

$\displaystyle\int_0^2 \frac{3}{2} y^3\, dy = \frac{3y^4}{8}\Big|_0^2 = 6$

6.



$[x^2 = 3x \Rightarrow x^2 - 3x = 0 \Rightarrow x=0 \text{ or } x=3]$

$\displaystyle\iint_D xy\, dx\, dy = -\int_0^3 \int_{3x}^{x^2} xy\, dy\, dx = -\int_0^3 x \frac{y^2}{2}\Big|_{3x}^{x^2} dx =$

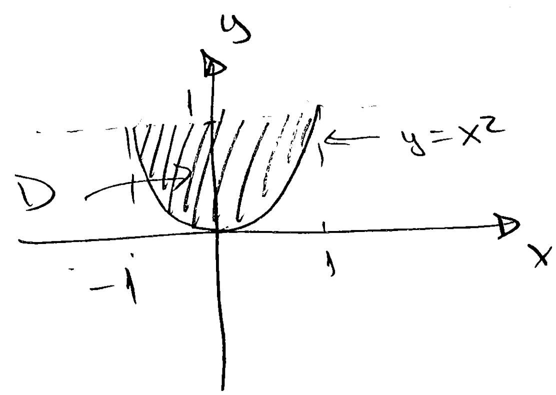
$\displaystyle -\int_0^3 \left(\frac{x^5}{2} - \frac{9x^3}{2}\right) dx = -\left(\frac{x^6}{12} - \frac{9x^4}{8}\right)\Big|_0^3 = -\left(\frac{729}{12} - \frac{729}{8}\right) = \frac{729}{24}$

$\displaystyle = \boxed{\frac{243}{8}}$

7. First we find the value of y at which the planes $3y$ and $2+y$ meet:

$$3y = 2+y \Rightarrow 2y = 2 \Rightarrow y = 1$$

which means that the region of integration is



$\therefore$

$$V = \iint_D (2+y)\,dA - \iint_D 3y\,dA$$

$$= \int_{-1}^{1}\int_{x^2}^{1} (2+y)\,dy\,dx - \int_{-1}^{1}\int_{x^2}^{1} 3y\,dy\,dx =$$

$$= \int_{-1}^{1} 2y + \frac{y^2}{2}\Big|_{x^2}^{1}\,dx - \int_{-1}^{1} \frac{3y^2}{2}\Big|_{x^2}^{1}\,dx =$$

$$= \int_{-1}^{1}\left[\left(2+\frac{1}{2}\right) - \left(2x^2 + \frac{x^4}{2}\right)\right]dx - \int_{-1}^{1}\left[\frac{3}{2} - \frac{3x^4}{2}\right]dx$$

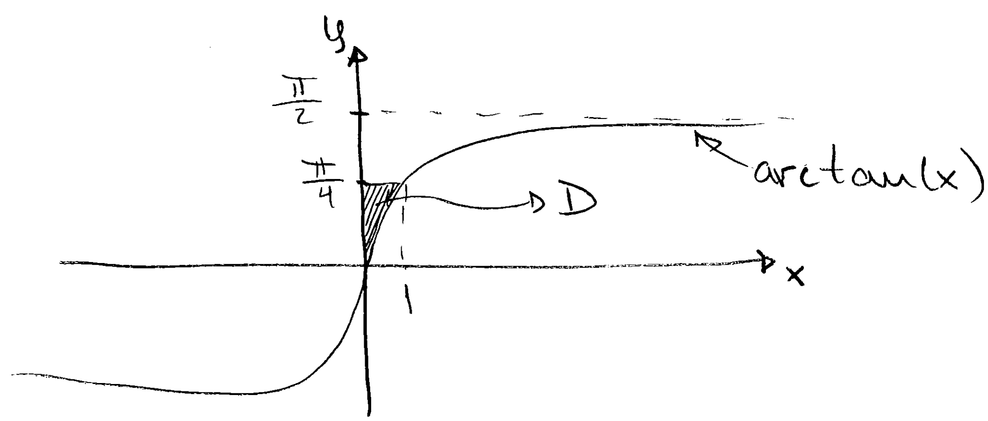
$$\left(\frac{5}{2}x - \frac{2}{3}x^3 - \frac{x^5}{10}\right)\Bigg|_{-1}^{1} - \left(\frac{3}{2}x - \frac{3x^5}{10}\right)\Bigg|_{-1}^{1} =$$

$$\left[\left(\frac{5}{2} - \frac{2}{3} - \frac{1}{10}\right) - \left(-\frac{5}{2} + \frac{2}{3} + \frac{1}{10}\right)\right] - \left[\left(\frac{3}{2} - \frac{3}{10}\right) - \left(-\frac{3}{2} + \frac{3}{10}\right)\right] =$$

$$\left(5 - \frac{4}{3} - \frac{2}{10}\right) - \left(3 - \frac{6}{10}\right) = 5 - 3 - \frac{2}{10} + \frac{6}{10} - \frac{4}{3} = 2 + \frac{4}{10} - \frac{4}{3} =$$

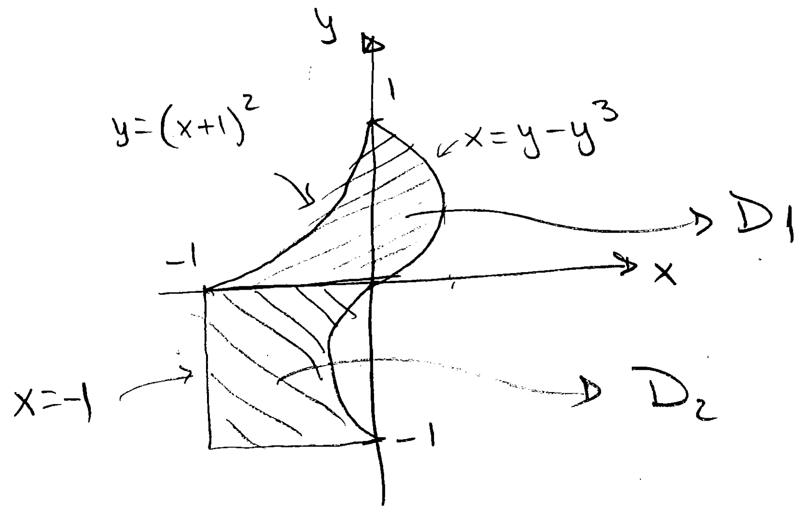
$$\frac{60+12-40}{30} = \frac{32}{30} = \boxed{\frac{16}{15}}$$

8.



$$\int_0^1 \int_{\arctan(x)}^{\pi/4} f(x,y)\,dy\,dx = \int_0^{\pi/4} \int_0^{\tan(y)} f(x,y)\,dx\,dy$$

9.



$D = D_1 \cup D_2$

$$\iint_D dA = \iint_{D_1} y\, dA + \iint_{D_2} y\, dA = \int_0^1 \int_{\sqrt{y}-1}^{y-y^3} y\, dx\, dy + \int_{-1}^{0} \int_{-1}^{y-y^3} y\, dx\, dy$$

$$\int_0^1 yx \Big|_{\sqrt{y}-1}^{y-y^3} dy + \int_{-1}^{0} yx \Big|_{-1}^{y-y^3} dy =$$

$$\int_0^1 y[(y-y^3) - (\sqrt{y}-1)]\, dy + \int_{-1}^{0} y[(y-y^3) - (-1)]\, dy =$$

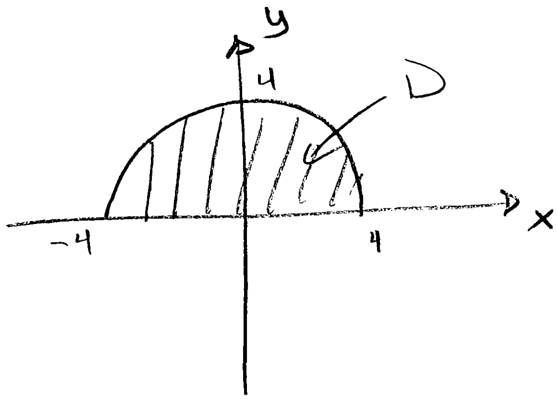
$$\int_0^1 (y^2 - y^4 - y^{3/2} + y)\, dy + \int_{-1}^{0} (y^2 - y^4 + y)\, dy =$$

$$\left(\frac{y^3}{3} - \frac{y^5}{5} - \frac{2}{5} y^{5/2} + \frac{y^2}{2}\right)\Bigg|_0^1 + \left(\frac{y^3}{3} - \frac{y^5}{5} + \frac{y^2}{2}\right)\Bigg|_{-1}^{0} =$$

$$\left(\frac{1}{3}-\frac{1}{5}-\frac{2}{5}+\frac{1}{2}\right)+\left(0-\left(\frac{-1}{3}+\frac{1}{5}+\frac{1}{2}\right)\right)=$$

$$\frac{1}{3}-\frac{3}{5}+\cancel{\frac{1}{2}}+\frac{1}{3}-\frac{1}{5}-\cancel{\frac{1}{2}}=\frac{2}{3}-\frac{4}{5}=\frac{10-12}{15}=\frac{-2}{15}$$

10.



$$\iint_D xy\,dA=\int_0^{\pi}\int_0^4 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta=$$

$$\int_0^{\pi}\int_0^4 r^3\cos\theta\sin\theta\,dr\,d\theta=\int_0^{\pi}\left.\frac{r^4}{4}\right|_0^4\cos\theta\sin\theta\,d\theta=$$

$$64\int_0^{\pi}\cos\theta\sin\theta\,d\theta\quad\left\{\begin{matrix}u=\sin\theta\\ du=\cos\theta\,d\theta\end{matrix}\right\}=$$

$$64\int_0^0 u\,du=\underline{0}$$