Homework #9

Math 243 - Section 51

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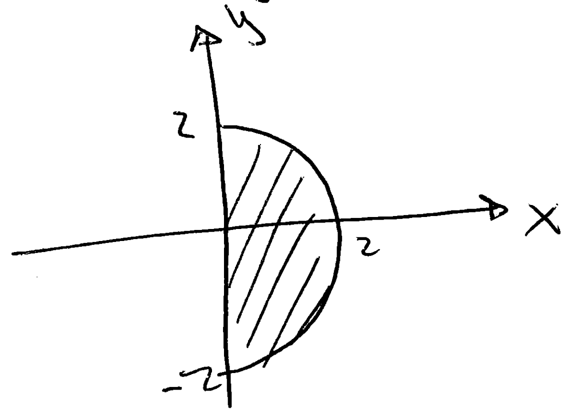
1. The region of integration is



Thus,

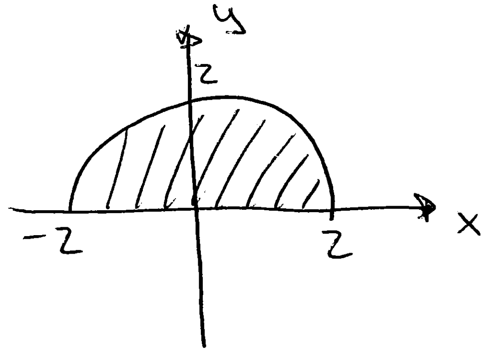
$$\iint_D e^{-(x^2+y^2)}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2} r\,dr\,d\theta \,; \quad \begin{aligned} u&=-r^2 \\ du&=-2r\,dr \end{aligned}$$

So,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 e^{-r^2} r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{-4} e^{u}\,du\,d\theta = -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{u}\Big|_0^{-4}\,d\theta =$$

$$-\frac{1}{2}\int_{-\frac{\pi}{2}}^{\pi/2}(e^{-4}-1)\,d\theta = -\frac{1}{2}(e^{-4}-1)\,\theta\Big|_{-\pi/2}^{\pi/2} = \boxed{-\frac{\pi}{2}(e^{-4}-1)}$$

2. The region of integration is



So
$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \sin(x^2+y^2)\,dy\,dx = \int_0^{\pi}\int_0^{2} \sin(r^2)\, r\,dr\,d\theta$$

$u = r^2$; $du = 2r\,dr \Rightarrow$

$$\int_0^{\pi}\int_0^{2} \sin(r^2)\,r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi}\int_0^{4} \sin(u)\,du\,d\theta = \frac{1}{2}\int_0^{\pi} -\cos(u)\Big|_0^4\,d\theta =$$

$$\frac{1}{2}\int_0^{\pi}(-\cos(4)+1)\,d\theta = \underline{\frac{\pi}{2}(1-\cos(4))}$$

3. $E = \{(x,y,z) \mid 1 \le z \le 4,\ z \le y \le 4,\ 0 \le x \le y\}$

$$\iiint_E \frac{y}{x^2+y^2}\,dV = \int_1^4\int_z^4\int_0^y \frac{y}{x^2+y^2}\,dx\,dy\,dz$$

using the formula $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$;

$$\int_1^4\int_z^4\int_0^y \frac{y}{x^2+y^2}\,dx\,dy\,dz = \int_1^4\int_z^4 y\cdot\left[\left(\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)\right)\right]_0^y dy\,dz =$$

$$\int_1^4\int_z^4 y\cdot\left(\frac{1}{y}\tan^{-1}(1)-\frac{1}{y}\tan^{-1}(0)\right)dy\,dz = \int_1^4\int_z^4 \frac{\pi}{4}\,dy\,dz =$$

$$\int_1^4 \frac{\pi}{4}y\Big|_z^4 dz = \int_1^4\left(\pi-\frac{\pi z}{4}\right)dz = \pi z - \frac{\pi z^2}{8}\Big|_1^4 =$$

$$4\pi - 2\pi - \left(\pi - \frac{\pi}{8}\right) = 2\pi - \frac{7\pi}{8} = \frac{16\pi - 7\pi}{8} = \underline{\frac{9\pi}{8}}$$

4.

$$\iiint_E xy\,dV = \int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{x+y} xy\,dz\,dy\,dx =$$

$$\int_0^1\int_{x^2}^{\sqrt{x}} xy(x+y)\,dy\,dx = \int_0^1\int_{x^2}^{\sqrt{x}}(x^2y+xy^2)\,dy\,dx =$$

$$\int_0^1\left(x^2\frac{y^2}{2}+x\frac{y^3}{3}\right)\Big|_{x^2}^{\sqrt{x}}dx = \int_0^1\left(\frac{x^3}{2}+\frac{x^{5/2}}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx =$$

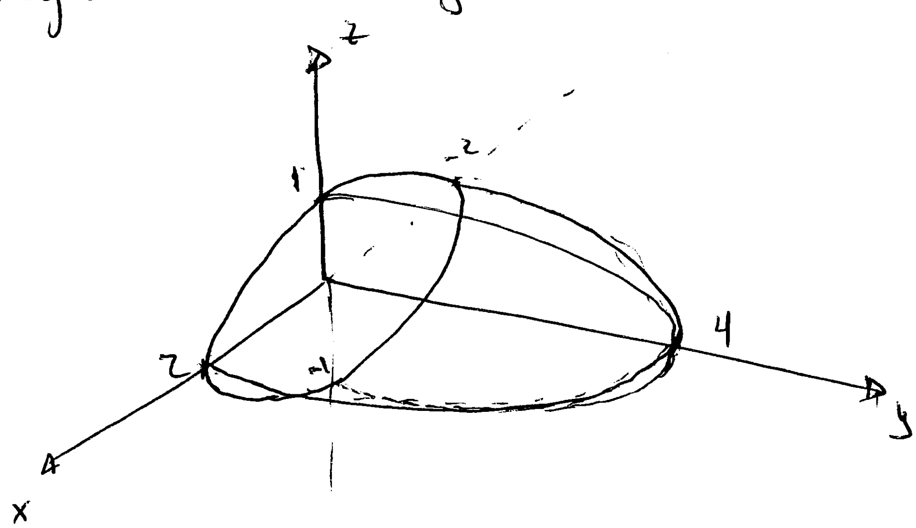
$$\left[\frac{x^4}{8} + \frac{2}{21}x^{7/2} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1 = \frac{1}{8} + \frac{2}{21} - \frac{1}{14} - \frac{1}{24} = \frac{3}{28}$$

5. $y = 4 - x^2 - 4z^2 \Rightarrow$ if $y = 0$ then

$$x^2 + 4z^2 = 4 \text{ or}$$

$$\left(\frac{x}{2}\right)^2 + z^2 = 1 \text{ (Ellipse)}$$

Thus, the region of integration looks like:



The different orders of integration are:

$$\int_{-1}^{1} \int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} \int_0^{4-x^2-4z^2} f(x,y,z)\, dy\, dx\, dz$$

$$\int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{4}}}^{\sqrt{\frac{4-x^2}{4}}} \int_0^{4-x^2-4z^2} f(x,y,z)\, dy\, dz\, dx$$

$$\int_0^4 \int_{-\sqrt{\frac{4-y}{4}}}^{\sqrt{\frac{4-y}{4}}} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\, dx\, dz\, dy$$

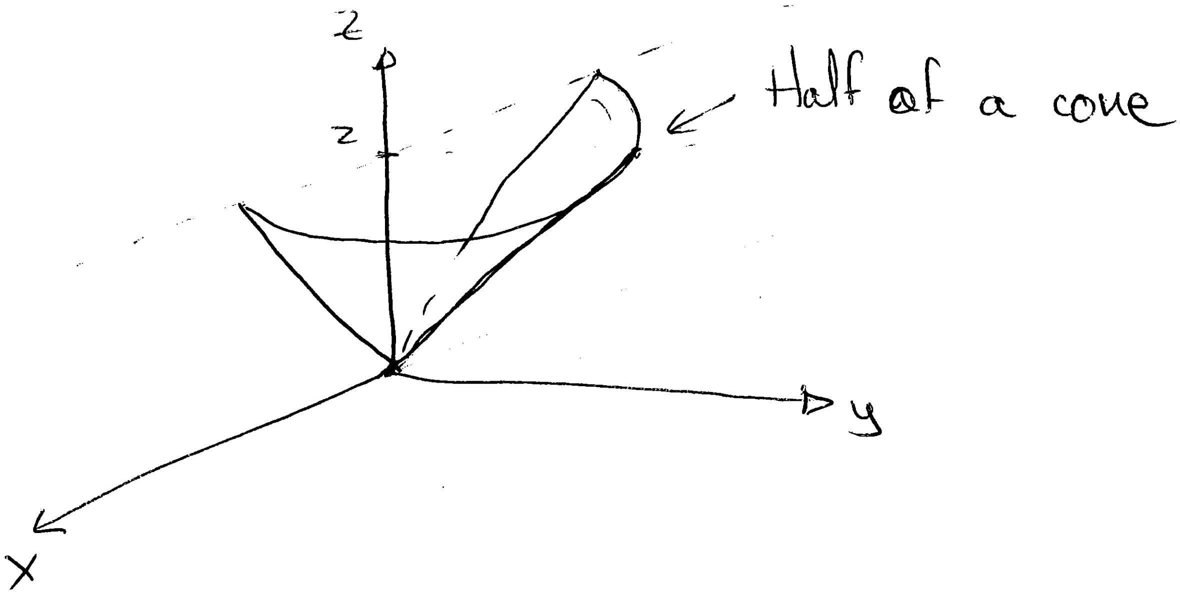
$$\int_{-1}^1 \int_0^{4-4z^2} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\, dx\, dy\, dz$$

$$\int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-\left(\sqrt{4-x^2-y}\right)/2}^{\left(\sqrt{4-x^2-y}\right)/2} f(x,y,z)\, dz\, dx\, dy$$

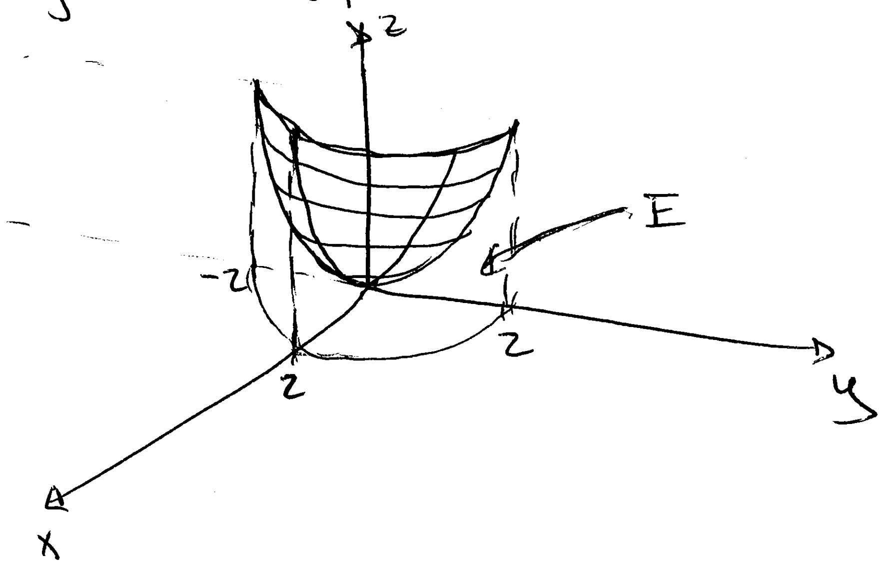
$$\int_{-2}^2 \int_0^{4-x^2} \int_{-\left(\sqrt{4-x^2-y}\right)/2}^{\left(\sqrt{4-x^2-y}\right)/2} f(x,y,z)\, dz\, dy\, dx$$

6. The integral will be maximum when $1-x^2-2y^2-3z^2 \geq 0$ because a triple integral is signed. Therefore, the region E is such that $x^2+2y^2+3z^2 \leq 1$, or it is the region bounded by the ellipsoid $\underline{x^2+2y^2+3z^2=1}$

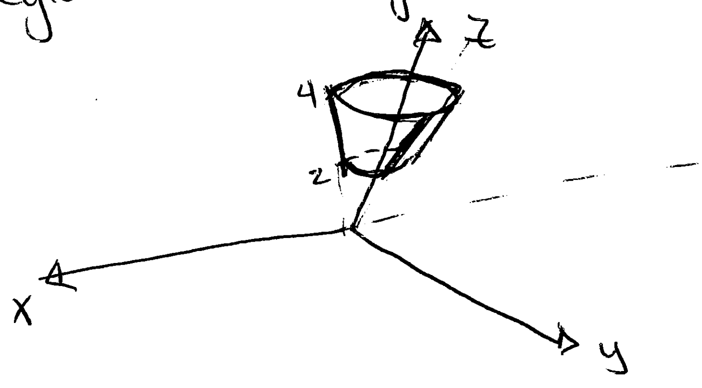
7.



8. The region of interest is the region bounded below by the xy-plane and the paraboloid:



9. The region of integration looks like



10.

From the equation of the sphere:

$$\rho^2 = x^2+y^2+z^2 = z = \rho\cos\phi \Rightarrow$$

$$\underline{\rho = \cos\phi}$$

From the equation of the cone

$$z = \rho\cos\phi = \rho\sin\phi = \sqrt{x^2+y^2} \Rightarrow$$

$$\tan\phi = 1 \Rightarrow \phi = \frac{\pi}{4}$$

$\therefore$ The solid in question can be represented by

$$0 \le \rho \le \cos\phi$$

$$0 \le \phi \le \frac{\pi}{4}$$

$$0 \le \theta \le 2\pi$$